Simultaneous state estimation and control for nonlinear systems subject to bounded disturbances ⋆

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Abstract

In this work, we address the output–feedback control problem for nonlinear systems under bounded disturbances using a moving horizon approach. The controller is posed as an optimization-based problem that simultaneously estimates the state trajectory and computes future control inputs. It minimizes a criterion that involves finite forward and backward horizon with respect the unknown initial state, measurement noises and control input variables and it is maximized with respect the unknown future disturbances. Although simultaneous state estimation and control approaches are already available in the literature, the novelty of this work relies on linking the lengths of the forward and backward windows with the closed-loop stability, assuming detectability and decoding sufficient conditions to assure system stabilizability. Simulation examples are carried out to compare the performance of simultaneous and independent estimation and control approaches as well as to show the effects of simultaneously solving the control and estimation problems.

Key words: Receding horizon control and estimation, Output feedback, Robust stability, Nonlinear systems.

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1 Introduction

One of the most popular control technique in both academia and industry is model predictive control (*MPC*) due to its ability to explicitly accommodate hard state and input constraints (Bemporad & Morari 1999, Camacho & Alba 2004, Rawlings & Mayne 2009, Mayne 2014, among others). Thereon, much effort has been devoted to developing a stability theory for *MPC* (see i.e. Rawlings & Mayne 2009, Grüne & Pannek 2011, Mayne 2016). An overview of recent developments can be found in Mayne (2014). *MPC* involves the solution of an open–loop optimal control problem at each sampling time with the current state as the initial condition. Each of these optimizations provides the sequences of future control actions and states. The first element of the control action sequence is applied to the system and, then the optimization problem is solved again at the next sampling time after updating the initial condition with the system state. *MPC* keeps constant the computational burden by optimizing the system behaviour within a finite length window. The system behaviour beyond the window is summarized in a term known as *cost–to–go*.

MPC is often formulated assuming that the system state can be measured. However, in many practical cases, the only information available is noisy measurements of system output, so the use of independent algorithms for state estimation (including observers, filters and estimators) becomes necessary (see Rawlings & Bakshi 2006). Of all these methods, moving horizon estimation (*MHE*) is especially engaging for use with *MPC* because it can be formulated as a similar online optimization problem. Solving the *MHE* problem produces an estimated state that is compatible with a set of past measurements that recedes as current time advances (Schweppe 1973, Rao et al. 2001, 2003). This estimate is optimal in the sense that it maximizes a criterion that captures the likelihood of the measurements. Along the same time that relevant results on *MPC* were developed, research works on *MHE* begun. The works of Rao et al. (2001) and (2003) provide overviews of linear and nonlinear *MHE*. Recent results regarding *MHE* for nonlinear systems are given for robust stability and estimate convergence properties (see Alessandri et al. 2005, 2008, 2012, Garcia-Tirado et al. 2016, Sánchez et al. 2017, among others). In recent years several results have been obtained for different *MHE* formulations, advancing from idealistic assumptions, like observability and vanishing disturbances, to realistic situations like detectability and bounded disturbances (see Ji et al. 2015, Müller 2017, Allan & Rawlings 2019, Deniz et al. 2019).

When disturbances, model uncertainty and system constraints can be neglected, state and control sequences can be independently computed (see Duncan & Varaiya 1971, Bensoussan 2004, Åström 2012, Georgiou & Lindquist 2013). However, in practical applications, these conditions are very difficult to fulfil, i.e., process disturbances and measurement noise are usually present, as well as model uncertainty. In this context, it becomes necessary approaches that include this information into the controller design. State-feedback *MPC* is a mature field with results that considers model uncertainty, input disturbances, and noises (Magni et al. 2003, Bemporad et al. 2003, Raimondo et al. 2009, among others). However, these works did not consider robustness with respect to errors in state estimation. Fewer results are available for output-feedback *MPC*. An overview of nonlinear output-feedback *MPC* is given by Findeisen et al. (2003) and the references therein. Many of these approaches involve designing separate estimator and controller, using different estimation algorithm (Roset et al. 2006, Magni et al. 2009, Patwardhan et al. 2012, Zhang & Liu 2013, Ellis et al. 2017). Results on robust output-feedback *MPC* for constrained, linear, discrete-time systems with bounded disturbances and measurement noise can be found in Mayne et al. (2006, 2009) and Voelker et al. (2010, 2013). These approaches first solve the estimation problem and prove the convergence of the estimated state to a bounded set, and then take the uncertainty of the estimation into account when solving the *MPC* problem.

The approach of solving simultaneously *MHE–MPC* was originally introduced by Copp & Hespanha (2014) and later developed in several papers (Copp & Hespanha 2016*a*,*b*, 2017). In the first paper, Copp & Hespanha (2014) proposed an output feedback controller that combines state estimation and control into a single $min-max$ optimization problem that, under observability and controllability assumptions (Copp & Hespanha 2016*a*), guarantees the boundedness of state and tracking errors. Finally, in the last work reported by Copp & Hespanha (2017), the authors established the conditions for guaranteeing the boundedness of error for trajectory tracking problems. They also introduced a primal–dual interior point method that can be used to efficiently solve the $min-max$ optimization problem. The criterion used in these works involves finite forward and backward horizons that are minimized with respect to feedback control policies and maximized with respect to the unknown parameters in order to guaranty robustness in the worst-case scenario.

In the present work, we introduce an output–feedback controller for nonlinear systems subject to bounded disturbances using simultaneous *MHE–MPC* approach. The resulting optimization problem minimizes a criterion that involves finite forward and backward horizons with respect the unknown initial state, measurement noise and control input variables while it is maximized with respect the unknown future disturbance variables. We show that the proposed controller results in closed–loop trajectories along which the states remain bounded. These results rely on two assumptions: The first assumption requires that the optimization criterion include an adaptive arrival cost (Sánchez et al. 2017). This assumption allows to ensure the boundedness of the state estimate and to obtain a bound for the estimation error set if the parameters of the estimation problem are properly chosen (Deniz et al. 2019). The second assumption requires that the backward (estimation) and forward (control) horizons are sufficiently large so that enough information is obtained in order to find state estimates and control inputs compatible with dynamics, noises and constraints. This assumption is satisfied if the system is detectable, stabilizable and the parameters in the cost function (weights and horizons) are chosen appropriately.

The rest of the paper is organized as follows: Section 2 introduces the notation, definitions and properties that will be used through the paper. In Section 3 we formulate the estimation and control problem, and in Section 4 we analyze its closed-loop stability. Section 5 discusses two examples to illustrate the concepts presented in this work. The first example uses a simple nonlinear model to analyse the consequences of simultaneously solving the estimation and control problems. The second example compares the performance obtained by the simultaneous and independent approaches applied to the regulation of the state of a van der Pol oscillator for two operational conditions. Finally, conclusion and future work is discussed in Section 6.

2 Preliminaries and setup

2.1 Notation

Let $\mathbb{Z}$ denotes the integer numbers, $\mathbb{Z}_{[a,b]}$ denotes the set of integers in the interval $[a,b]$, with $b>a$ and $\mathbb{Z}_{\geq a}$ denotes the set of integers greater or equal to a. Boldface symbols denote sequences of finite ($\boldsymbol{w} := \{w_1, \ldots, w_2\}$) or infinite ($\boldsymbol{w} := \{w_1, \ldots, w_2, \ldots\}$) length. We denote $\hat{x}_{j|k}$ as the state at time j estimated at time k. By $|x|$ we denote the euclidean norm of a vector $x \in \mathbb{R}^{n_x}$. Let $\|\boldsymbol{x}\| := \sup_{k\in\mathbb{Z}_{\geq 0}} |x_k|$ denote the supreme norm of the sequence $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_{[a,b]} := \sup_{k\in\mathbb{Z}_{[a,b]}} |x_k|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ if γ is continuous, strictly increasing and $\gamma(0) = 0$. If γ is also unbounded, it is of class $\mathscr{K}_\infty$. A function $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{L}$ if ζ is continuous, non increasing and $\lim_{t\to\infty} \zeta(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{KL}$ if $\beta(\cdot, k)$ is of class $\mathscr{K}$ for each fixed $k \in \mathbb{Z}_{\geq 0}$, and $\beta(r, \cdot)$ of class $\mathscr{L}$ for each fixed $r \in \mathbb{R}_{\geq 0}$. Let us consider now two sets A and B, the Minkowski addition is defined as $A \oplus B := \{a+b|\, a\in A, b \in B\}$. On the other hand, the Minkowski difference [1] is defined as $A \ominus B := \{d|\, d+b \in A\}$. In the following sections, we will use the notation $\Psi_{p,t,l}$ to reference the cost incurred solving the problem p at time t with a horizon length l, while $\Psi_{p,t,l}(x)$ will be used to indicate the cost at the solution x, with x belonging to a consistent domain with the cost function $\Psi_{p,t,l}$. When necessary, we will use the notation $x^{(1)}_{i,k}$ and $x^{(2)}_{i,k}$ to differentiate i–th component of the state vector of two discrete-time

[1] Also known as the Pontryagin difference.

trajectories of the system, with $i \in \mathbb{Z}_{[1,n]}$. Moreover, $x_k^{(1)}(x_0^{(1)}, \boldsymbol{w}^{(1)})$ will denote a trajectory with initial condition $x_0^{(1)}$ and perturbed by the sequence $\boldsymbol{w}^{(1)}$. A similar notation is used for the case of continuous time systems, where t is used instead k to denote continuous time.

2.2 Problem statement

Let us consider a discrete-time nonlinear system whose behaviour is given

$$
\begin{aligned}
x_{k+1} &= f\left(x_k, u_k\right) + w_k \quad \forall\, k \in \mathbb{Z}_{\geq 0}, \\
y_k &= h\left(x_k\right) + v_k,
\end{aligned}
\tag{1}
$$

where $x \in \mathscr{X} \subseteq \mathbb{R}^{n_x}$ is the system state, $u \in \mathscr{U} \subseteq \mathbb{R}^{n_u}$ is the system's input and $w \in \mathscr{W} \subseteq \mathbb{R}^{n_w}$ is the unmeasured process disturbance posed as an additive input. The output of the system is $y \in \mathscr{Y} \subseteq \mathbb{R}^{n_y}$ and $v \in \mathscr{V} \subseteq \mathbb{R}^{n_v}$ is the measurement noise. The function $f(\cdot,\cdot)$ is assumed to be at least locally Lipschitz in its arguments, and the function $h(\cdot)$ is known to be a continuous function. The sets $\mathscr{X}$, $\mathscr{U}$, $\mathscr{W}$, $\mathscr{V}$ and $\mathscr{Y}$ are assumed to be convex, containing the origin in its interior. The estimation and control problem attempts to simultaneously find the optimal state $\hat{x}_{k|k}$ and the optimal sequence of control inputs $\hat{\boldsymbol{u}}$ which will steer the system to the desired operation zone. It is in an infinite-horizon optimization problem given by

$$
\begin{aligned}
\min_{\hat{x}_{0|k}, \hat{\boldsymbol{w}}, \hat{\boldsymbol{u}}} \Psi_{EC,k,\infty} &:= \sum_{j=0}^{k} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) + \sum_{j=k}^{\infty}\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) \\
\text{s.t.} & \begin{cases}
\hat{x}_{j+1|k} = f\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) + \hat{w}_{j|k}, \\
y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \\
\hat{x}_{j|k} \in \mathscr{X},\ \hat{u}_{j|k} \in \mathscr{U}, \hat{w}_{j|k} \in \mathscr{W},\ \hat{v}_{j|k} \in \mathscr{V}.
\end{cases}
\end{aligned}
\tag{2}
$$

Functions $\ell_e(\hat{w}_{j|k}, \hat{v}_{j|k})$ penalize large values of $\hat{w}_{j|k}$ and $\hat{v}_{j|k}$, whereas $\ell_c(\hat{x}_{j|k}, \hat{u}_{j|k})$ penalize large values of the predicted state $\hat{x}_{j|k}$ and control inputs $\hat{u}_{j|k}$. The function $\ell_{w_c}(\hat{w}_{j|k})$ is assumed to take non–negative values and since it is subtracting in the objective function, process disturbances will tend to be maximized within the control window. When necessary, we will decompose the function $\ell_e(\cdot,\cdot)$ into $\ell_{w_e}(\cdot)$ and $\ell_{v_e}(\cdot)$ which penalizes $\hat{w}_{j|k}$ and $\hat{v}_{j|k}$, respectively. Problem (2) is valuable from a theoretical point of view since it guarantees the boundedness of the estimates $\hat{x}_{j|k}$ and control actions $\hat{u}_{j|k}$ provided the cost function is bounded, i.e., $\Psi_{EC,k,\infty} \leq \gamma$, $\forall\, k \in \mathbb{Z}_{\geq 0}$, with $\gamma \in \mathbb{R}_{\geq 0}$. If functions $\ell_e(\cdot,\cdot)$, $\ell_c(\cdot,\cdot)$ and $\ell_{w_c}(\cdot)$ are defined using a norm–ℓ_p, problem (2) would guarantee that the state x_k and u_k are ℓ_p , provided that noises w_k and v_k are also ℓ_p. This would mean that the closed-loop system has a finite ℓ_p–induced gain.

The infinite–horizon problem (2) lacks practical interest since it is intractable from a computational point of view. Then, it is reformulated into a receding finite–horizon problem

$$
\begin{aligned}
\min_{\hat{x}_{k-N_e|k},\hat{\boldsymbol{w}},\hat{\boldsymbol{u}}} \Psi_{EC,k,N_e+N_c} &:= \Gamma_{k-N_e}(\chi) + \sum_{j=k-N_e}^{k} \ell_e\left(\hat{w}_{j|k},\hat{v}_{j|k}\right) + \\
&\sum_{j=k}^{k+N_c-1}\left(\ell_c\left(\hat{x}_{j|k},\hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}(\Xi) \\
\text{s.t.} &\begin{cases}
\chi = \hat{x}_{k-N_e|k} - \bar{x}_{k-N_e}, \\
\Xi = \hat{x}_{k+N_c|k}, \\
\hat{x}_{j+1|k} = f\left(\hat{x}_{j|k},\hat{u}_{j|k}\right) + \hat{w}_{j|k}, \\
y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \\
\hat{x}_{j|k} \in \mathscr{X}, \Xi \in \mathscr{X}_f \subseteq \mathscr{X},\ \hat{u}_{j|k} \in \mathscr{U}, \hat{w}_{j|k} \in \mathscr{W},\ \hat{v}_{j|k} \in \mathscr{V}.
\end{cases}
\end{aligned}
\tag{3}
$$

For computation tractability, the infinite summations of $\Psi_{EC,k,\infty}$ have been replaced by backward and forward windows of finite length, corresponding to the estimation Ψ_{E,k,N_e} and control Ψ_{C,k,N_c} problems of criterion Ψ_{EC,k,N_e+N_c}, respectively. Ψ_{E,k,N_e} includes N_e terms backward in time from sample k corresponding to the *estimator stage-cost*, $\ell_e\left(\hat{w}_{j|k},\hat{v}_{j|k}\right)$, and the extra term $\Gamma_{k-N_e}(\chi)$, known as *arrival-cost*, that summarizes information left behind the estimation window by penalizing the uncertainty in the initial state $\hat{x}_{k-N_e|k}$ (Rao et al. 2001, 2003). On the other hand, Ψ_{C,k,N_c} includes N_c terms forward in time from sample k corresponding to the *controller stage-cost*, $\ell_c(\hat{x}_{j|k},\hat{u}_{j|k}) - \ell_{w_c}(\hat{w}_{j|k})$, and an extra term $\Upsilon_{k+N_c}(\Xi)$, known as *cost-to-go*, that summarizes the behaviour of the system beyond the control window by penalizing the deviation of the final state $\Xi = \hat{x}_{k+N_c|k}$. Moreover, the set $\mathscr{X}_f$ represent the set of terminal constraints, as common in *MPC* (Rawlings et al. 2017).

The goal of problem (3) is to estimate the initial state $\hat{x}_{k-N_e|k}$ and disturbances $\hat{w}_{j|k}$ $j \in \mathbb{Z}_{[k-N_e,k-1]}$ such that an estimate $\hat{x}_{k|k}$ is obtained to compute the control inputs $u_{j|k}$ $j \in \mathbb{Z}_{[k,k+N_c-1]}$ that drive the system states to the desired region. Therefore, there is no point on penalizing the control cost $\ell_c(\cdot,\cdot)$ along the estimation window. The variables $\hat{v}_{j|k}$ are not independent variables since they are uniquely determined by the remaining optimization variables and the output equation

$$\hat{v}_{j|k} = y_j - h(\hat{x}_{j|k}), \qquad j \in \mathbb{Z}_{[k-N_e,k]}. \tag{4}$$

Since there is no measurement of future system output, $\hat{v}_{j|k}$ will not be considered along the control window. However, the disturbances $\hat{w}_{j|k}$ needs to be considered along both windows Ψ_{E,k,N_e} and Ψ_{C,k,N_c} because they affect all the states, starting from $j = k - N_e - 1$. As will be shown later, the ratio between disturbances w_j and control actions u_j, for $j \in \mathbb{Z}_{[k,k+N_c-1]}$, encodes the controllability property of the system, imposing a bound on the relation between w_j and u_j in order to avoid to lose system controllability. However, in practical implementations, the process disturbance variables $\hat{w}_{j|k}$ along the control horizon can be omitted to avoid increase the computational burden of the optimization problem.

Remark 1 *The sequence of process disturbances $\hat{w}_{j|k}$ is minimized within the estimator window, i.e., $j \in [k - N_e - 1, k - 1]$, and it is maximized within the controller window, $j \in [k, k + N_c - 1]$.*

2.3 Relationship with MHE and MPC

The criterion Ψ_{EC,k,N_e+N_c} can be rewritten as follows

$$\Psi_{EC,k,N_e+N_c} \coloneqq \varphi\Psi_{E,k,N_e} + (1-\varphi)\Psi_{C,k,N_c}, \quad \varphi \in [0,1], \tag{5}$$

where Ψ_{E,k,N_e} is the criterion implemented by a *MHE* estimator and Ψ_{C,k,N_c} is to the criterion implemented by a *robust MPC* controller, given by

$$\begin{aligned} \Psi_{E,k,N_e} &\coloneqq \Gamma_{k-N_e}(\chi) + \sum_{j=k-N_e}^{k} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right), \\ \Psi_{C,k,N_c} &\coloneqq \Upsilon_{k+N_c}(\Xi) + \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right). \end{aligned} \tag{6}$$

Equation (5) corresponds to a weighted sum multi-objective formulation of criterion (3), where φ controls the influence of Ψ_{E,k,N_e} on Ψ_{C,k,N_c}. When $\varphi = 0$, $\Psi_{EC,k,N_e+N_c} \coloneqq \Psi_{C,k,N_c}$ and problem (3) becomes a *robust model predictive control* problem with terminal cost considered by Chen & Allgöwer (1998), given that x_k is measurable or it is provided by an estimator. On the other case, when $\varphi = 1$, $\Psi_{EC,k,N_e+N_c} \coloneqq \Psi_{E,k,N_e}$ and problem (3) becomes a *moving horizon estimation* problem considered by Ji et al. (2016), Garcia-Tirado et al.

(2016), Müller (2017), Deniz et al. (2019), given that the control inputs $u_{j|k}$ are computed by a controller. In these cases, the optimization problem (3) has only one objective and the separation principle needs to be applied since the estimator and the controller are implemented independently.

When $0 < \varphi < 1$, Ψ_{E,k,N_e} and Ψ_{C,k,N_c} are simultaneously considered by Ψ_{EC,k,N_e+N_c} and the optimization problem (3) becomes multi-objective. The importance of Ψ_{E,k,N_e}, and therefore the one of Ψ_{C,k,N_c}, is defined by φ emphasizing or deemphasizing the influence of the estimation problem on the solution. In the case of $\varphi = 0.5$, Ψ_{E,k,N_e} and Ψ_{C,k,N_c} have similar influence on the solution of (3) and it becomes the problem proposed by Copp & Hespanha (2017).

Definition 1 *Let assume points* $z_E \in \mathbb{R}^{n_w N_e} \times \mathbb{R}^{n_v(N_e+1)} \times \mathbb{R}^{n_x(N_e+1)} =: \mathscr{Z}_E$ *and* $z_C \in \mathbb{R}^{n_w N_c} \times \mathbb{R}^{n_u N_c} \times \mathbb{R}^{n_x(N_c+1)} =: \mathscr{Z}_C$ *such that* $z \in \mathscr{Z}_E \times \mathscr{Z}_C =: \mathscr{Z}$. *A point* $z^o \in \mathscr{Z}$, *is Pareto optimal iff there does not exist another point* $z \in \mathscr{Z}$ *such that* $\Psi_{EC,N_e+N_c,k}(z) \leq \Psi_{EC,N_e+N_c,k}(z^o)$ *and* $\Psi_{E,N_e,k}(z_E) < \Psi_{E,N_e,k}(z_E^o)$, $\Psi_{C,N_c,k}(z_C) < \Psi_{C,N_c,k}(z_C^o)$ *(Miettinen 2012).*

According to this concept, problem (3) looks for solutions that neither $\Psi_{E,N_e,k}$ nor $\Psi_{C,N_c,k}$ can be improved without deteriorate one of them. Any optimal solution of problem (3) with $0 < \varphi < 1$ is Pareto optimal (Miettinen 2012), therefore it has an optimal trade-off between $\Psi_{E,N_e,k}$ and $\Psi_{C,N_c,k}$. On the other cases, $\varphi = 0$ or $\varphi = 1$ the solutions of problem (3) are optimal in the sense of the selected objective ($\Psi_{E,N_e,k}$ or $\Psi_{C,N_c,k}$, respectively). In these cases, the solutions obtained are not Pareto optimal and, therefore the overall system performance can be poorer than the one provided by the multi-objective problem.

From a practical point of view, φ can be used to improve the numerical properties of the optimization problem (3). This fact allows to improve the convergence properties of the numerical algorithms employed to solve it (see Example 4.2). For example, if $N_e \ll N_c$ and the stage costs $\ell_e(\cdot), \ell_c(\cdot)$ and $\ell_{w_c}(\cdot)$ have similar values, the optimization problem will improve $\Psi_{C,N_c,k}$ at the expense of $\Psi_{E,N_e,k}$ (because $\Psi_{C,N_c,k} \gg \Psi_{E,N_e,k}$), deteriorating the estimation of $\hat{x}_{k|k}$ and producing potentially ill conditioned Jacobian and Hessian matrices of Ψ_{EC,k,N_e+N_c}. This numerical problems can lead to an increment of the computational times of the optimization problem. A similar situation can happen when $N_e \gg N_c$.

3 Robust stability under bounded disturbances

In this section, we introduce the results regarding feasibility and robust stability of the proposed algorithm. Firstly, the properties of *MHE* and *MPC* are analyzed and then the results for the simultaneous *MHE–MPC* are given. Besides, feasibility conditions for the existence of a solution to (3) and minimum horizon lengths required to achieve the desired estimation and control performances are analyzed.

3.1 Backward window

The simultaneous state estimation and control problem relies on a backward window of fixed length N_e to compute the optimal state estimate $\hat{x}_{k|k}$. Then, the controller takes the estimate $\hat{x}_{k|k}$ as initial condition and predicts the system behaviour. To take advantage of the backward window and reconstruct the state of the system, there have to exists an observer for it, i.e., the system has to be detectable. A definition of detectability for nonlinear systems is *incremental input-output-to-state stability -i-IOSS-* (Sontag & Wang 1995), and it entails that the difference between any two trajectories of the system can be bounded by

$$\begin{aligned} |x_k(x_0^{(1)}, \boldsymbol{w}^{(1)}) - x_k(x_0^{(2)}, \boldsymbol{w}^{(2)})| \leq \beta\left(|x_0^{(1)} - x_0^{(2)}|, k\right) + \gamma_1\left(\|\boldsymbol{w}^{(1)} - \boldsymbol{w}^{(2)}\|\right) \\ + \gamma_2\left(\|h\left(\boldsymbol{x}^{(1)}\right) - h\left(\boldsymbol{x}^{(2)}\right)\|\right), \end{aligned} \tag{7}$$

with $\beta(\cdot,\cdot) \in \mathscr{K}\mathscr{L}$, $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathscr{K}$. In the following, we assume that the system is *i-IOSS*, i.e., any two trajectories eventually become indistinguishable one of another. Note that inequality (7) only includes the process disturbance as input to the system. For the case of non-autonomous system, as in the present work, control inputs also have to be taken into account. Since control inputs and process disturbances have the same nature in our context, considering both is straightforward. Moreover, as will be shown later in Example 4.1, the control law chosen have not only effects in the forward window but also in the backward window influencing on the estimation process.

Previous results on robust output-feedback *MPC* with bounded disturbances firstly solve the estimation problem and show the convergence of estimated states to a bounded set, then take the uncertainty of estimation into account when solving the *MPC* problem (Mayne et al. 2006, 2009). The key idea in these works was to consider the estimation error as an additional, unknown but bounded uncertainty that must be accounted for guaranteeing stability and feasibility of the resulting closed–loop system. Let us define the *robust estimable set*

$$\mathscr{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e(k)\right) := \left\{x : |x - \hat{x}_{k|k}| \leq \varepsilon_e(k), \forall\, \hat{x}_{k|k}\right\} \tag{8}$$

where $\hat{x}_{k|k}$ is the best estimate available and ε_e is the estimation error at time k bounded by (Deniz et al. 2019)

$$\varepsilon_e(k) \leq \bar{\Phi}\left(|x_0 - \bar{x}_0|, k\right) + \pi_w\left(\|\boldsymbol{w}\|\right) + \pi_v\left(\|\boldsymbol{v}\|\right). \tag{9}$$

Functions $\bar{\Phi}$, π_w and π_v are defined in term of *MHE* parameters as follows

$$\begin{aligned}
\bar{\Phi}\left(|x_0 - \bar{x}_0|, k\right) &\coloneqq \theta^i |x_0 - \bar{x}_0|^\zeta \frac{\mathbb{N}_e}{N_e} \left(\left(\frac{\overline{\lambda}_{P^{-1}}}{\underline{\lambda}_{P^{-1}}} \right)^\rho (c_\beta 18^p + \right. \\
&\left. \underline{\lambda}_{P^{-1}}^{\alpha_1} \left(P^{-1}_{k-N_e}\right) \left(c_1\, 3^{\alpha_1} + c_2\, 3^{\alpha_2}\right)\Big) + c_\beta\, 2^p \right), && (10) \\
\pi_w\left(\|\boldsymbol{w}\|\right) &\coloneqq 2\left(1+\mu\right) \left(\frac{c_\beta\, 18^p}{\underline{\lambda}_{P^{-1}}}\, \bar{\gamma}_w^{\frac{p}{a}}\left(\|\boldsymbol{w}\|\right) + c_2\, 3^{\alpha_2} \bar{\gamma}_w^{\alpha_2}\left(\|\boldsymbol{w}\|\right) + \right. \\
&\left. \gamma_1\left(6\|\boldsymbol{w}\|\right) + \gamma_1\left(6 \underline{\gamma}_w^{-1}\left(3\bar{\gamma}_w\left(\|\boldsymbol{w}\|\right)\right)\right) \right), && (11) \\
\pi_v\left(\|\boldsymbol{v}\|\right) &\coloneqq 2\left(1+\mu\right) \left(\frac{c_\beta\, 18^p}{\underline{\lambda}_{P^{-1}}}\, \bar{\gamma}_v^{\frac{p}{a}}\left(\|\boldsymbol{v}\|\right) + c_1\, 3^{\alpha_1} \bar{\gamma}_v^{\alpha_1}\left(\|\boldsymbol{v}\|\right) + \right. \\
&\left. \gamma_2\left(6\|\boldsymbol{v}\|\right) + \gamma_2\left(6 \underline{\gamma}_v^{-1}\left(3\bar{\gamma}_v\left(\|\boldsymbol{v}\|\right)\right)\right) \right), && (12)
\end{aligned}$$

where $\theta = \frac{2+\mu}{2(1+\mu)} < 1$, $\mu \in \mathbb{R}_{\geq 0}$, $i = \lfloor \frac{k}{N_e} \rfloor$, $\underline{\lambda}_{P^{-1}}$ and $\overline{\lambda}_{P^{-1}}$ are the minimal and maximal eigenvalues of the arrival-cost weight matrix P, respectively. Moreover, the matrix P is updated at each sampling time applying the algorithm developed in Sánchez et al. (2017). As in the case of the stage cost, the arrival–cost is lower and upper bounded by

$$\underline{\lambda}_{P^{-1}} |\chi|^2 \leq \Gamma_{k-N_e}\left(\chi\right) \leq \overline{\lambda}_{P^{-1}} |\chi|^2. \tag{13}$$

On the other hand, ζ, ρ, c_β, p, a, c_1, c_2, α_1 and α_2 are positive real constants whose value depend on the system and parameters of the estimator (Deniz et al. 2019). The functions γ_1 and γ_2 are related with the system detectability (equation (7)), whereas the functions γ_w and γ_v are bounds of the stage-cost of the estimator, whose relationship is given by

$$\begin{aligned}
\underline{\gamma}_w\left(|\hat{w}_{j|k}|\right) &\leq \ell_{w_e}\left(\hat{w}_{j|k}\right) \leq \overline{\gamma}_w\left(|\hat{w}_{j|k}|\right), \\
\underline{\gamma}_v\left(|\hat{v}_{j|k}|\right) &\leq \ell_{v_e}\left(\hat{v}_{j|k}\right) \leq \overline{\gamma}_v\left(|\hat{v}_{j|k}|\right),
\end{aligned} \tag{14}$$

and N_e is the length of the backward window. $\mathbb{N}_e$ is the minimum length of the backward window required to guarantee the boundness of the estimation error, which is given by

$$\mathbb{N}_e = \left\lceil \left(2^{\zeta} e_{max}^{\zeta-1} \bar{c}_{\beta}\right)^{\frac{1}{\eta}} \right\rceil, \tag{15}$$

where $e_{\max}$ denotes the maximal error on the prior estimate of the initial condition and $\eta \in \mathbb{R}_{\geq 0}$ is a constant. Henceforth, we will assume that $N_e \geq \mathbb{N}_e$.

At each sampling time, the measurements available along the backward window are used to obtain $\hat{x}_{k|k}$. Whenever $N_e \geq \mathbb{N}_e$, the estimation error will decrease until it reaches an invariant space whose volume depends on the process and measurement noises as well as the stage- cost and the system itself. The behaviour of the system is forecast from the estimate $\hat{x}_{k|k}$, whereas x_k remains within $\mathscr{E}_{N_e}$.

3.2 Forward window

The forward window corresponds to the *MPC* problem, which computes the optimal control inputs $\hat{\boldsymbol{u}}$ using $\hat{x}_{k|k}$ as initial condition. Its feasibility depends on the fact that its initial condition x_k must belong to the *robust controllable set* $\mathscr{R}_{N_c}(\Omega, \mathbb{T})$ (Kerrigan & Maciejowski 2000), which is defined as follows

$$\mathscr{R}_{N_c}(\Omega, \mathbb{T}) \coloneqq \left\{ x_0 \in \Omega | \exists\, u_j \in \mathscr{U} \ : \{x_j \in \Omega, x_{N_c} \in \mathbb{T}\} \quad \forall j \in \mathbb{Z}_{[0, N_c-1]} \right\}. \tag{16}$$

Since $x_k \in \mathscr{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e\right)$ the feasibility of the control problem is guaranteed if $\mathscr{E}_{N_e}\left(\hat{x}_{k|k}, \varepsilon_e\right) \subseteq \mathscr{R}_N(\Omega, \mathbb{T}) \quad \forall k \geq 0$, which implies $\mathscr{X}_f \subseteq \mathbb{T}$. Note that this feasibility condition is not only necessary for the simultaneous *MHE–MPC*, but also for independent *MHE* and *MPC* (Mayne et al. 2006, 2009). Let us state this condition in the following assumption

Assumption 1 *The robust estimable set $\mathscr{E}_{N_e}$ belong to the robust controllable set $\mathscr{R}_N(\Omega, \mathbb{T})$ in N_c steps for all times $k \geq 0$*

$$\mathscr{E}_{N_e} \subseteq \Omega,\ \mathscr{X}_f \subseteq \mathbb{T} \rightarrow \mathscr{R}_{N_c}(\mathscr{E}_{N_e}, \mathscr{X}_f) \quad \forall k \geq 0. \tag{17}$$

This assumption states that despite the sequence of control is computed from an estimate $\hat{x}_{k|k}$, provided that x_k belong to $\mathscr{R}_{N_c}(\mathscr{E}_{N_e}, \mathscr{X}_f)$, $x_{k+1} \in \mathscr{R}_{N_c}(\mathscr{E}_{N_e}, \mathscr{X}_f)$. Moreover, the volume of the robust estimable set decrease faster with longer backward windows and the size of the robust controllable set can be enlarged by mean of larger forward window and with the appropriate design of the set $\mathscr{U}$.

Regarding stability along the forward window, a common approach to guarantee the stability of *MPC* is by mean of the inclusion of a terminal constraint set, which is generally a level set of a control Lyapunov function (Mayne et al. 2000). This set is an artificial constraint set but guarantees stability (Tuna et al. 2006). In this work we will analyse the stability of the controller following a similar approach as in Tuna et al. (2006), where the analysis is carried out as a function of the length of the forward window, taking into account the effect of the process disturbances and the estimation errors. A pseudo measure of the system controllability property will be introduced and the minimum forward window length which guarantees the stability of the simultaneous *MHE–MPC* is given, without imposing extra terminal constraints nor appeal for the cost-to-go to be a closed–loop Lyapunov function (*CLF*). In this sense, let us state the following assumption.

Assumption 2 *There exist a constant $\delta \in \mathbb{R}_{\geq 0}$ such that the cost-to-go and the stage cost satisfy the following relation:*

$$\Upsilon_{k+N_c}\left(f\left(x,u\right)\right) - \Upsilon_{k+N_c}\left(x\right) \leq -\ell_c\left(x,u\right) + \Upsilon_{k+N_c}\left(x\right)\delta + \ell_{w_c}\left(w\right). \tag{18}$$

A similar assumption was already used in Tuna et al. (2006), where the constant δ is introduced in order to relax the requirement on function $\Upsilon_{k+N_c}(\cdot)$ to be a *CLF* for the nominal case. Despite we use a different notation for the cost-to-go term $\Upsilon_{k+N_c}(\Xi)$, this function can take the same behaviour as the stage-cost, i.e., $\Upsilon_{k+N_c}(\Xi) = \ell_c(\Xi, 0)$. Here we extend it to the more general case where process disturbances are affecting the system, and it will lead, as will be shown later, in longer control windows. However, in practical implementation, one can omit process disturbance optimization variables to avoid increasing the computational burden but setting the length of the forward window to the value computed with the process disturbance taken into account.

Regarding the elements of the optimization problem corresponding to the control problem, we will assume that the stage-cost is lower bounded.

Assumption 3 *The stage cost $\ell_c(x,u)$ is lower bounded by a function $\sigma(x) \in \mathscr{K}_\infty$, such that $\sigma(x) \leq \ell_c(x,u) \quad \forall x \in \mathscr{X}, u \in \mathscr{U}$.*

Note that for a quadratic stage-cost, i.e., $\ell_c(x,u) = x^T Q x + u^T R u$, with Q and R positive definite matrices, one can choose $\sigma(x) = \underline{\lambda}_Q |x|^2$, where $\underline{\lambda}_Q$ denotes the minimal eigenvalue of matrix Q. Moreover, we will assume that there exist an increasing sequence that relates the function $\sigma(x)$ with the cost of the control problem $\Psi_{C,k,i}$, where i represents different lengths of the forward window.

Assumption 4 *There exists a sequence* $\boldsymbol{L} \coloneqq [L_0, L_1, \ldots, L_j]$, $L_i \in \mathbb{R}$, $1 \leq L_i \leq L$, $i \in \mathbb{Z}_{\geq 0}$ *that verifies*

$$\Psi_{C,k,i} \leq L_i \, \sigma\left(x\right). \tag{19}$$

Choosing

$$L_i = \frac{\Psi_{C,k,i}}{\sigma(x)}, \tag{20}$$

satisfies inequality (19) even for $L_0 = 1$, since $\Psi_{C,k,0} = \Upsilon_k(\Xi) = \sigma(x)$. Finally, let us define the following quantity

$$\Delta_c^w \coloneqq \max\left\{ \min_{\hat{u}_{k|k}} \frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} \right\}, \, \forall\, \hat{x}_{k|k} \in \mathscr{X}, \, \forall\, \hat{w}_{k|k} \in \mathscr{W}. \tag{21}$$

It encodes a pseudo–measure of the system controllability relating the capability of control actions to compensate the process disturbances. The term pseudo–measure is used here because the relation Δ_c^w is given via the penalization functions $\ell_{w_c}(\cdot)$ and $\ell_c(\cdot,\cdot)$. In the following, we will assume that the system is controllable from this point of view.

Assumption 5 *The controller of the system can be designed such that the following relation can always be verified*

$$\Delta_c^w < 1 \tag{22}$$

With the properties established for the backward and forward windows in mind, next we will study the overall stability of the simultaneous *MHE–MPC*.

3.3 Backward and forward windows

With all the elements introduced in the previous section, we are ready to derive the main result: the stability of the resulting closed-loop system of the proposed output-feedback controller with estimation horizon $\mathbb{N}_e$ and control horizon $\mathbb{N}_c$ for nonlinear detectable and controllable systems under bounded disturbances.

Theorem 1 *Given the i-IOSS nonlinear system* (1) *with a prior estimate* $\bar{x}_0 \in \mathscr{X}_0$ *of its unknown initial condition* x_0 *and bounded disturbances* $\boldsymbol{w} \in \mathscr{W}(w_{\max})$, $\boldsymbol{v} \in \mathscr{V}(v_{\max})$, *Assumptions 1 to 5 are fulfilled, the estimation window verifies* $N_e \geq \mathbb{N}_e$ *and the control horizon* N_c *verifies*

$$\mathbb{N}_c = \left\lceil 1 + \frac{\ln\left(\frac{\delta(L-1)}{1-\Delta_c^w}\right)}{\ln\left(\frac{L}{L-1}\right)} \right\rceil, \tag{23}$$

then there will exist at each sampling time k *a feasible estimate* $\hat{x}_{k-N_e|k}$ *and feasible sequences* $\hat{\boldsymbol{w}}$ *and* $\hat{\boldsymbol{u}}$ *such that*

$$\Delta\Psi \leq -\,\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)(1-\delta\omega) + \overline{\pi}_E, \tag{24}$$

where

$$\begin{aligned} \omega &:= \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\delta}\,\Delta_c^w, \qquad 0 \leq \delta\omega < 1 \\ \overline{\pi}_E &:= \overline{\gamma}_w\left(\underline{\gamma}_w^{-1}\left(\frac{\overline{\gamma}_p(\chi)}{N_e} + \overline{\gamma}_w(\|\boldsymbol{w}\|) + \overline{\gamma}_v(\|\boldsymbol{v}\|)\right)\right). \end{aligned} \tag{25}$$

Proof. See Appendix A. $\square$

4 Examples

In this section, we discuss two examples to illustrate the results presented previously and compare the performance of the framework discussed formerly. The first example applies the ideas introduced in previous sections to a nonlinear scalar system. The emphasis is placed in the effect of constraints and disturbances on closed-loop stability and performance. The second example discusses the simulations results for a van der Pol oscillator using the framework discussed in previous sections. The discussion is focused on the effect of N_e and N_c on the performance and computational time.

4.1 Example 1

Let us consider the continuous–time nonlinear scalar system

$$
\begin{aligned}
\dot{x} &= a x_t^3 + w_t + u_t, \quad a \in \mathbb{R}_{>0} \\
y_t &= x_t + v_t.
\end{aligned}
\tag{26}
$$

Firstly, we show its detectability, i.e., the existence of an estimate with a structure like equation (7). Let assume two arbitrary and feasible trajectories $x_t^{(1)}$ and $x_t^{(2)}$ such that $\Delta x \coloneqq x_t^{(1)} - x_t^{(2)}$ and $p_t \coloneqq |\Delta x|$; then $\dot{p}_t$ can be written as follows

$$
\dot{p}_t = \frac{\Delta x}{|\Delta x|} \left(\dot{x}_t^{(1)} - \dot{x}_t^{(2)} \right). \tag{27}
$$

Assuming a LTV control law $u_t = -K_t x_t$, we obtain

$$
\dot{p}_t = \frac{\Delta x}{|\Delta x|} \left(a \Delta x \left(x_t^{(1)^2} + x_t^{(1)} x_t^{(2)} + x_t^{(2)^2} \right) - K_t \Delta x + \Delta w_t \right), \tag{28}
$$

which is upper bounded by

$$
\dot{p}_t \leq -K_t \, p_t + a \, g |\Delta h_t| + |\Delta w_t|, \tag{29}
$$

where

$$
g \coloneqq h^2 \left(x_0^{(1)} \right) + h \left(x_0^{(1)} \right) h \left(x_0^{(2)} \right) + h^2 \left(x_0^{(2)} \right). \tag{30}
$$

Solving p_t for initial condition $p_0 = |x_0^{(1)} - x_0^{(2)}|$ we obtain

$$
|x_t^{(1)} - x_t^{(2)}| \leq \; |x_0^{(1)} - x_0^{(2)}| e^{-K_t t} + \frac{\|\Delta w_{0:t}\|}{K_t} + \frac{a g \|\Delta y_{0:t}\|}{K_t}, \tag{31}
$$

it follows the fact that system (26) is *i-IOSS* (for all details, the reader can refer to appendix B).

In the case of *MHE–MPC* controllers with quadratic costs

$$\begin{aligned} \ell_e &:= \hat{w}_{j|k}^2 Q_e + \hat{v}_{j|k}^2 R_e,\ \Gamma_{k-N_e} := P_{k-N_e}^{-1}\chi^2, \\ \ell_c &:= \hat{x}_{j|k}^2 Q_c + \hat{u}_{j|k}^2 R_c,\ \Upsilon_{k+N_c} := S_c \Xi^2, \end{aligned} \tag{32}$$

analysed in this work, the bound (31) can be written as follows

$$\begin{aligned} |x_k - \hat{x}_{k|k}| \leq\ & |x_0 - \bar{x}_0| \left(\tfrac{\theta\,\mathbb{N}_e}{2N_e}\right)^i \left(2 + \frac{\left(P_{k-N_e}^{-1} R_e\right)^{1/2} + \left(P_{k-N_e}^{-1} Q_e\right)^{1/2} a\,g}{(Q_e R_e)^{1/2} K_{1|k}^c}\right) \\ & + 2\,(1+\mu)\,\|\boldsymbol{w}\| \left(\frac{2}{K_{1|k}^c} + \frac{(Q_e\,R_e)^{1/2} K_{1|k}^c + \left(P_{k-N_e}^{-1}\,Q_e\right)^{1/2} a\,g}{\left(P_{k-N_e}^{-1}\,Q_e\right)^{1/2} K_{1|k}^c}\right) \\ & + 2\,(1+\mu)\,\|\boldsymbol{v}\| \left(\frac{2\,g}{K_{1|k}^c} + \frac{(R_e\,Q_e)^{1/2} K_{1|k}^c + \left(P_{k-N_e}^{-1}\,R_e\right)^{1/2}}{\left(P_{k-N_e}^{-1} Q_e\right)^{1/2} K_{1|k}^c}\right), \end{aligned} \tag{33}$$

with $\mathbb{N}_e$ given by

$$\mathbb{N}_e = \left\lceil 4\left(2 + \frac{\left(P_{k-N_e}^{-1}\,R_e\right)^{1/2} + \left(P_{k-N_e}^{-1}\,Q_e\right)^{1/2} a\,g}{(Q_e\,R_e)^{1/2}\,K_{1|k}^c}\right)^2\right\rceil, \tag{34}$$

and $K_{1|k}^c$ is the equivalent controller gain resulting from applying $\hat{u}_{1|k}$.

Equations (33) and (34) show the influence of the controller on the state estimation. Larger controller gains improve estimation error and shorten the convergence time. However, controller gains are bounded by robust stability conditions and input constraints, which are limiting factors in this potential improvement. This example highlights the relevance of simultaneously solving the estimation and control problems, or at least to take into account the solution of control problem on the estimation one. Since *MPC* gains $K_{1|k}^c$ are time-varying because they are recomputed at every sampling time, a conservative approach can employ its lowest value.

In order to compare the performances of independent and simultaneous *MHE–MPC*, both output–feedback controllers have the same parameters

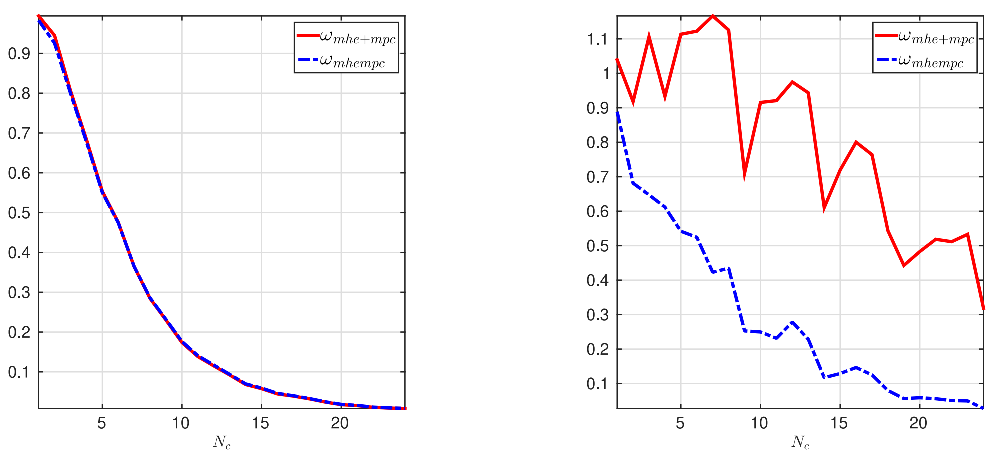
$$P_0 = 10^5,\ Q_e = 15,\ R_e = 10^3,\ Q_c = 5,\ R_c = 5,\ S_c = Q_c,\ \mu = 0.05, \tag{35}$$

with constraints sets

$$\mathscr{X} \coloneqq \{x : |x| \leq 0.8\} \text{ and } \mathscr{U} \coloneqq \{u : |u| \leq 2.5\}, \tag{36}$$

$w_t \sim U\,(0,\, 0.01)$, $v_t \sim \mathscr{N}\,(0,\, 0.02^2)$, $a = 1$, $g = 3x^3_{0\,\max}$ and $\varphi = 0.5$ such that both controllers implement the same optimization criterion.

The control problems of both controllers are configured without terminal constraints. The process disturbance is not taken into account to compute $\hat{u}_{j|k}$, but it will be considered in the computation of N_c. It can be computed directly from equation (23) once the values of δ, L and Δ^w_c had been established. Another approach, employed in this example, consists of computing ω through simulations. In this example, we set the initial condition that maximizes the controller costs and then computes the values of L and ω. The process is repeated until reach the maximal value of N_c.

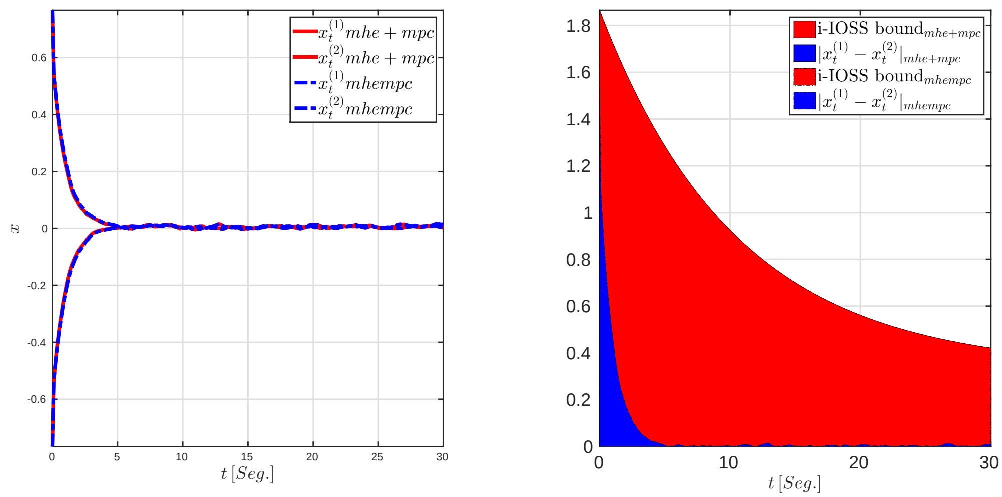


(a) (b)

Fig. 1. $\omega(N_c)$ for $\delta = 1, \Delta^w_c = 10^{-1}$ and constraints sets (36) (1a) and (37) (1b) for independent (red) and simultaneous (blue) approaches.

Figure 1 shows the computed values of ω a funtion of N_c ($\omega(N_c)$) for the same Δ^w_c and different set of constraints and distributions for process and measurement noises. In this figure the effect of constraints on $\omega(N_c)$ can be seen: They increase $\omega(N_c)$, for the same N_c, depending how the controller is implemented. This change is smaller for the simultaneous *MHE–MPC* approach than the independent one. When constraints are no relevant (constraints set (36)), both controllers have similar values (see Figure 1a), and the control problem of both controllers can use the same N_c. However when constraints are tighten (constraints set (37)), the way of solving the estimation and control problems has a direct effect on $\omega(N_c)$ (see Figure 1b), and the control problem of both controllers must use different N_c in order to ensure robust stability, affecting the computational requirements of the implementation. Since we are using constraints set (36) we choose $N_c = 10$ for both controllers (Figure 1a).

Finally, the minimum estimation horizon $\mathbb{N}_e$ is computed from (15) using the parameters listed in (35), leading to $\mathbb{N}_e = 27$ for both controllers. We choose $N_e = 30$ for both controllers.



(a) (b)

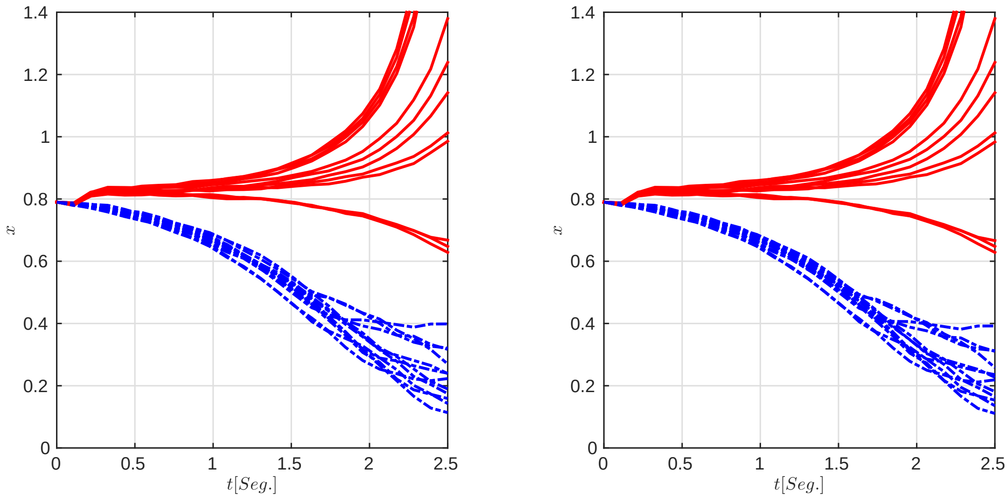
Fig. 2. Evolution of system output for different initial conditions, difference between trajectories and *i-IOSS* bound.

Figure 2 shows the system responses and the corresponding *i-IOSS* bound for the regulation problem. Figure 2a shows two trajectories generated by both controllers from different initial condition ($x_0^{(1)} = 0.766$ and $x_0^{(2)} = -0.766$) with the same prior guess ($\bar{x}_0 = -2.5$). Figure 2b shows the difference between the trajectories and its *i-IOSS* bound, for the minimum controller gain along the simulation ($K^c_{1|k} = 0.7326$). One can see in this figure the decreasing behaviour of the estimation error bound, as expected from equation (9) for the general case and (33) for this particular example. Despite the small value of μ ($\mu = 0.05$), the bound (33) is quite conservative. In these figures, we can also see that both controllers provide a similar response, since constraints and disturbances have not relevant effect on the system behaviour, and therefore the separation principle can be applied.

Now let us compare the performance in a more challenging setup. In the following, we will assume the next constraints set

$$\mathscr{U} := \{u : |u| \leq 0.6\}\,, \mathscr{W} := \{w : |w| \leq 0.4\} \text{ and } \mathscr{V} := \{v : |v| \leq 0.8\}\,. \quad (37)$$

The controls $\hat{u}_{j|k}$ have been tightened and the estimates $\hat{w}$ and $\hat{v}$ have been constrained to the sets $\mathscr{W}$ and $\mathscr{V}$, respectively. Disturbances w_t and v_t are now given by $w_t \sim U\,(0,\,0.1)$ and $v_t \sim \mathscr{N}\,(0,\,0.2^2)$, respectively.



(a) (b)

Fig. 3. Evolution of system output for $N_c = 20$ (3a) and $N_c = 70$ (3b), with $N_e = 30$ for independent *MHE* and *MPC* (red line) and simultaneous *MHE–MPC* (blue dotted line).

Under this new operational conditions $\mathbb{N}_e$ is recomputed, obtaining $\mathbb{N}_e = 98$ for the independent *MHE* and *MPC*, and $\mathbb{N}_e = 52$ for the simultaneous *MHE–MPC*. This is the effect of constraints set (37) on the estimator parameters, while the effect on the controller is shown in Figure 1b. This figure shows that the independent *MHE* and *MPC* approach is more sensitive to disturbances, requiring conservative values of N_c to guarantee the closed–loop stability.

Finally, Figures 3 show the system responses for regulation problem for different realizations of w_t and v_t and different N_c, for $N_e = 30 < \mathbb{N}_e$. The independent *MHE* and *MPC* strategy fails to regulate the system states for some noise realizations, even though it regulates few of them. On the other hand, the simultaneous *MHE–MPC* controller manages to regulate the system states for all noise realizations. This problem is caused by the failure of the independent *MHE* and *MPC* to satisfy Assumption 1. In fact, its design procedure applies the *separation principle*, which entails the automatic satisfaction of Assumption 1 and it does not include the constraints information in the selection of N_e and N_c. On the other hand, the simultaneous *MHE–MPC* controller does.

4.2 Example 2

Let us consider the van der Pol oscillator whose dynamic is described by

$$\dot{x}_t = \begin{bmatrix} \epsilon\left(1 - x_{2,t}^2\right) x_{1,t} - 2x_{2,t} + u_t + w_{1,t} \\ 2x_{1,t} + w_{2,t} \end{bmatrix} \quad \epsilon \in \mathbb{R}_{\geq 0}, \tag{38}$$
$$y_t = \tfrac{1}{2}\left(x_{1,t} + x_{2,t}\right) + v_{1,t}.$$

It is known to be *i-IOSS*, and a proof of this property can be made using the averaging lemma (Pogromsky & Matveev 2015).

In this example we will focus the analysis on the system performance under different set of parameters. The independent and simultaneous *MHE–MPC* controllers have the same parameters to allow a direct comparison of their performances. All the stage costs have a quadratic structure (equation (32)) and their parameters are

$$P_0 = 10^5,\ Q_e = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix},\ R_e = 150,\ Q_c = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix},\ S = Q_c,\ R_u = 10^{-2},$$

with constraints sets given by

$$\mathscr{X} \coloneqq \{x : |x_1| \leq 5, |x_2| \leq 5\},\ \mathscr{U} \coloneqq \{u : |u| \leq 5, |\Delta u_k| \leq 2\}, \tag{39}$$

$w_t \sim U(0, 0.25)$ and $v_t \sim U(0, 0.025)$, instead of zero mean normal distribution, as it is common in the literature.

The effect of N_e and N_c on closed-loop performance is be analysed for the following values

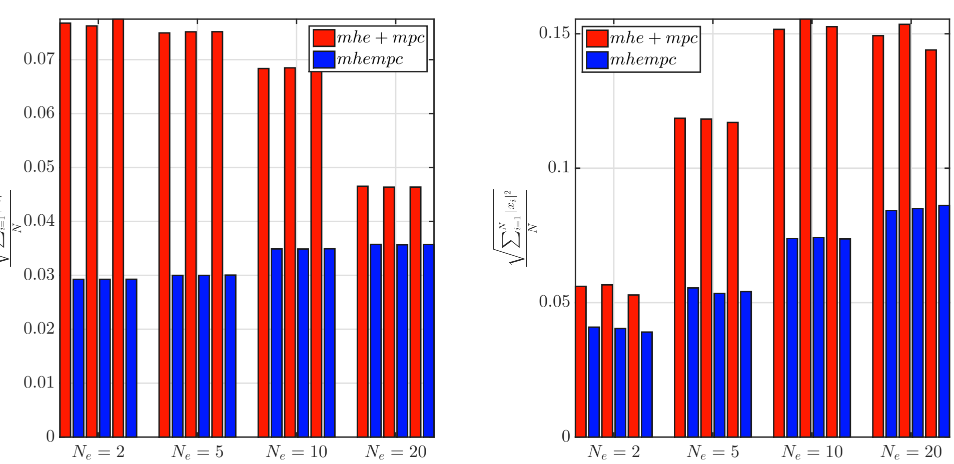
$$N_e \coloneqq \{2, 5, 10, 20\},\ N_c \coloneqq \{5, 10, 35\}. \tag{40}$$

Since the difference between N_e and N_c can lead to unbalanced cost functions (emphasizing the control cost over the estimation one), which can deteriorate the overall closed-loop performance. To avoid this problem, φ is used to improve the closed-loop performance. It takes the following values $\varphi \coloneqq \{0.95, 0.95, 0.85, 0.65\}$ for the corresponding N_e value.

Figures 4 summarize the mean square error (*MSE*) obtained by both controllers along 100 simulations for $\epsilon = 0.1$ and $\epsilon = 3$ respectively. These figures show the superior performance of the simultaneous *MHE–MPC* for any combination of $N_e - N_c$ and scenario. In general, there are no meaningful changes of MSE with N_c, however closed-loop performance varies with N_e. Figure 4a shows the results for $\epsilon = 0.1$. In this case the independent MHE and MPC performance improves with N_e, while the simultaneous MHE–MPC ones remains similar (a deviation lower than 8% from the average) for any combination of $N_e - N_c$. For this value of ϵ, the system (38) behaves like a harmonic oscillator, therefore the closed-loop performance depends on the estimation error (see Figure 5), which decreases for larger values of N_e. Figure 4b shows the results for $\epsilon = 3$. In this condition, the performance of both controllers deteriorates with N_e, because for this value of ϵ the system (38) behaves like a non-linear dampened oscillator and the state estimates take longer to converge to the estimation invariant set (see Figures 5a and 5b).

Figures 5 show the simulations resulting from two noise realizations for $N_c = 35$, $N_e = 2$ and $N_e = 20$, respectively. They show that the simultaneous MHE–MPC manages to regulate both states and it achieves a better performance than the independent one. While Figures 5a and 5c show that independent MHE and MPC achieves a better performance than the simultaneous one for state x_1, Figures 5b and 5d show how it fails to regulate state x_2 for short estimation horizons. Under this condition, x_2 has an offset that it is not compensated by the controller. Only large values of N_e allow the independent MHE and MPC to regulate x_2 (Figure 5d). On the other hand, the simultaneous MHE–MPC regulates both states and it takes shorter times than the independent one to regulate both states.

The computational burden of the simultaneous MHE–MPC is lower than the independent one, as can be seen in Figure 6. The execution times were averaged over 100 trials. The lower time, in the beginning, is due to the backward window corresponding to the estimation has not achieved yet its full length.



(a) (b)

Fig. 4. MSE of 100 simulations for different values of N_e, N_c and ϵ

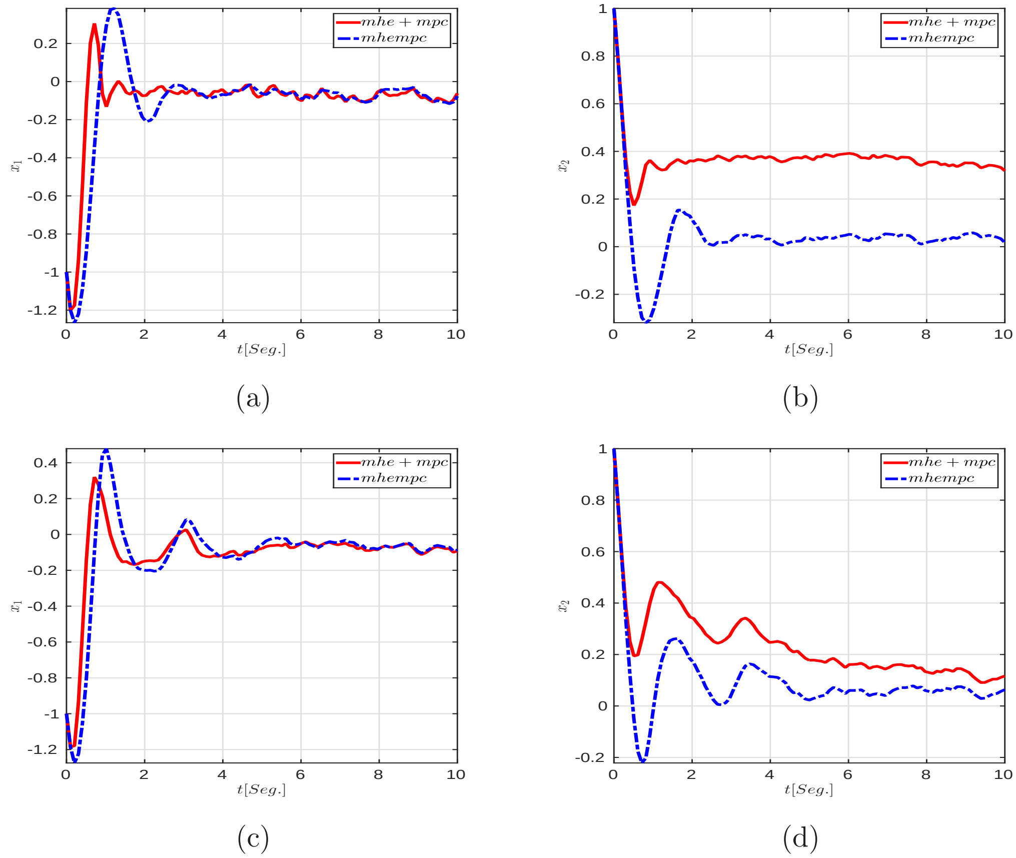


Fig. 5. Two realizations of x_1 and x_2 for $\epsilon = 0.1$, $N_e = 2$ (5a - 5b), $N_e = 20$ (5c - 5d) and $N_c = 35$.

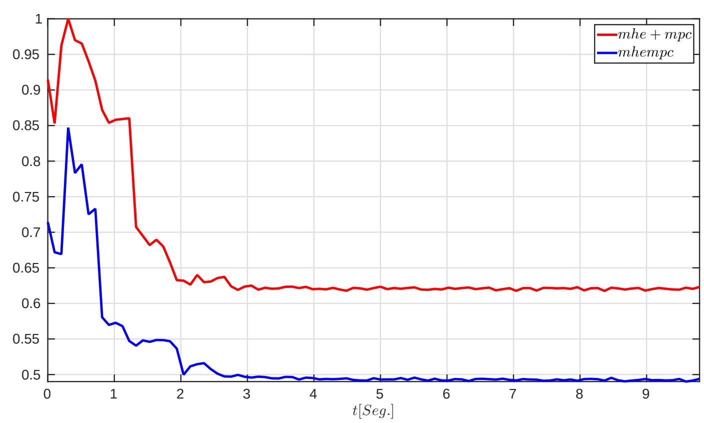


Fig. 6. Average execution over 100 trials for $N_e = N_c = 10$.

5 Conclusions

We presented an output-feedback approach for nonlinear systems subject to bounded disturbances using *MHE–MPC*. The proposed approach combines the state estimation and control problems into a single optimization, which is solved at each sampling time. Theorem 1 states the necessary conditions to guaranty the feasibility and stability of the optimization problem, and therefore the boundedness of system states, as a function of the windows lengths N_e and N_c. This result requires the compatibility between the robust estimated and controllable sets (Assumption 1) and the existence of a relaxed closed–loop Lyapunov function for the disturbed system (Assumption 18). These conditions imply forward (N_c) and backward (N_e) horizons to find state estimates and control actions that are consistent with the system dynamics, constraints and disturbances. Future work may involve the design of the forward window with properties that allow the improvement of the estimation process and the design of an adaptive law to compute φ such that the estimation and control problems keep balanced and the overall system performance and numerical properties are improved.

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A Proof Theorem 1

In the following we will analyse the stability of the simultaneous *MHE–MPC* algorithm by means of the difference in costs at two consecutive sampling time

$$\Delta\Psi = \Psi_{EC,k+1,N_e+N_c} - \Psi_{EC,k,N_e+N_c}. \tag{A.1}$$

Evaluating $\Psi_{EC,k+1,N_e+N_c}$ with the tail of the solution computed at time k, with $\hat{u}_{k+N_c} = 0$ and $\hat{x}_{k+N_c+1} = f(\Xi, \hat{u}_{k+N_c})$, we obtain

$$\begin{aligned}
\Delta\Psi = {}& \Gamma_{k-N_e+1}\left(\chi_{k-N_e+1}\right) + \sum_{j=k-N_e+1}^{k} \ell_{w_e}\left(\hat{w}_{j|k+1}\right) + \sum_{j=k-N_e+1}^{k+1} \ell_{v_e}\left(\hat{v}_{j|k+1}\right) \\
&+ \sum_{j=k+1}^{k+N_c} \left(\ell_c\left(\hat{x}_{j|k+1}, \hat{u}_{j|k+1}\right) - \ell_{w_c}\left(\hat{w}_{j|k+1}\right)\right) + \Upsilon_{k+N_c+1}\left(f(\Xi, \hat{u}_{k+N_c})\right) \\
&- \left(\Gamma_{k-N_e}\left(\chi\right) + \sum_{j=k-N_e}^{k-1} \ell_{w_e}\left(\hat{w}_{j|k}\right) + \sum_{j=k-N_e}^{k} \ell_{v_e}\left(\hat{v}_{j|k}\right) \right.\\
&\left. + \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Upsilon_{k+N_c}\left(\Xi\right)\right).
\end{aligned} \tag{A.2}$$

Since $\chi_{k-N_e+1} = \hat{x}_{k-N_e+1|k+1} - \bar{x}_{k-N_e+1}$ and

$$\begin{aligned}
\bar{x}_{k-N_e+1} &= \hat{x}_{k-N_e+1|k}, \\
\hat{x}_{k-N_e+1|k+1} &= \hat{x}_{k-N_e+1|k},
\end{aligned} \tag{A.3}$$

then $\Gamma_{k-N_e+1}(\chi_{k-N_e+1}) = 0$. Using inequality (18) and Assumption 5, $\Delta\Psi$ can be rewritten as follows

$$\begin{aligned}
\Delta\Psi \leq {}& -\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)\left(1 - \delta\left(\frac{\Upsilon_{k+N_c}\left(\Xi\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\delta}\frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}\right)\right) \\
& -\Gamma_{k-N_e}\left(\chi\right) + \ell_{w_e}\left(\hat{w}_{k|k+1}\right) - \ell_{w_e}\left(\hat{w}_{k-N_e|k}\right) - \ell_{v_e}\left(\hat{v}_{k-N_e|k}\right), \\
\leq {}& -\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)\left(1 - \delta\left(\frac{\Upsilon_{k+N_c}\left(\Xi\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\delta}\Delta_c^w\right)\right) \\
& -\Gamma_{k-N_e}\left(\chi\right) + \ell_{w_e}\left(\hat{w}_{k|k}\right) - \ell_e\left(\hat{w}_{k-N_e|k}, \hat{v}_{k-N_e|k}\right).
\end{aligned} \tag{A.4}$$

for $\delta \in \mathbb{R}_{\geq 0}$. Defining functions ω and π_E as follows

$$\begin{aligned} \omega &:= \frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\delta}\,\Delta_c^w, \\ \pi_E &:= -\Gamma_{k-N_e}(\chi) + \ell_{w_e}\left(\hat{w}_{k|k}\right) - \ell_e\left(\hat{w}_{k-N_e|k}, \hat{v}_{k-N_e|k}\right), \end{aligned} \tag{A.5}$$

the equation (A.4) can be written in a compact way

$$\Delta\Psi \leq -\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)(1-\delta\omega) + \pi_E. \tag{A.6}$$

The term ω quantifies the improvements in the control cost (through the ratio between the *cost-to-go* $\Upsilon_{k+N_c}(\cdot)$ and the control stage cost $\ell_c(\cdot,\cdot)$ at time k) and the disturbance controllability (the ratio between the control stage costs $\ell_{w_c}(\cdot)$ and $\ell_c(\cdot,\cdot)$ at time k).

The term π_E quantifies the changes in the estimation cost by measuring the amount of information left behind the estimation window (the *arrival–cost* $\Gamma_{k-N_e}(\cdot)$). Since $\hat{w}_{k|k}$ was computed within the control window (maximized), it tends to take larger values than $\hat{w}_{k-N_e|k}$ which was computed within the estimation window (minimized). Therefore, when state estimation is precise (i.e., $\Gamma_{k-N_e}(\chi)$ remains low), the term π_E will tend to take positive values, whereas if a major correction is made on the initial condition $\hat{x}_{k-N_e|k}$ (i.e., $\Gamma_{k-N_e}(\chi)$ will take big values), the improvement in the estimated trajectory will lead a decreasing cost with sharper slope.

Since

$$\begin{aligned} \pi_E &= -\Gamma_{k-N_e}(\chi) + \ell_{w_e}\left(\hat{w}_{k|k}\right) - \ell_e\left(\hat{w}_{k-N_e|k}, \hat{v}_{k-N_e|k}\right), \\ &\leq \ell_{w_e}\left(\hat{w}_{k|k}\right), \\ &\leq \overline{\gamma}_{w_e}\left(|\hat{w}_{k|k}|\right), \\ &\leq \overline{\gamma}_{w_e}\left(\|\hat{\boldsymbol{w}}\|\right), \end{aligned} \tag{A.7}$$

which can be written in term of $\mathscr{K}$ functions as follows (Deniz et al. 2019)

$$\pi_E \leq \overline{\gamma}_w\left(\|\hat{\boldsymbol{w}}\|\right) \leq \overline{\pi}_E := \overline{\gamma}_w\left(\underline{\gamma}_w^{-1}\left(\frac{\overline{\gamma}_p(\chi)}{N_e} + \overline{\gamma}_w(\|\boldsymbol{w}\|) + \overline{\gamma}_v(\|\boldsymbol{v}\|)\right)\right), \tag{A.8}$$

Restating (A.6) with $\overline{\pi}_E$, $\Delta\Psi$ can be posed as

$$\Delta\Psi \leq -\,\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)(1-\delta\omega)+\overline{\pi}_E, \tag{A.9}$$

From the first term in the right hand side of (A.9), one can see that if

$$0 \leq \delta\omega < 1 \tag{A.10}$$

then, for large values of $\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)$ so that it becomes dominating in (A.9), the sequence of cost will present a contractive behaviour until $\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)(1-\delta\omega)$ reaches the value of $\overline{\pi}_E$. Therefore, we are looking for a control horizon large enough such that

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)} < \frac{1-\Delta_c^w}{\delta} \tag{A.11}$$

Since $\Delta_c^w < 1$ by assumption 5, right hand side of inequality (A.11) will be positive. The problem consists now in to find a value of N_c such that (A.11) be verified. In order to relate (A.11) with N_c, let us note that

$$\begin{aligned}\Psi_{C,k,N_c} &= \textstyle\sum_{j=k}^{k+N_c-1}\left(\ell_c\left(\hat{x}_{j|k},\hat{u}_{j|k}\right)-\ell_{w_c}\left(\hat{w}_{j|k}\right)\right)+\Upsilon_{k+N_c}(\Xi),\\ &= \ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)\textstyle\sum_{j=k}^{k+N_c-1}\frac{\left(\ell_c\left(\hat{x}_{j|k},\hat{u}_{j|k}\right)-\ell_{w_c}\left(\hat{w}_{j|k}\right)\right)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}+\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)},\\ &\leq \ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)\textstyle\sum_{j=k}^{k+N_c-1}\frac{\left(\ell_c\hat{x}_{j|k},\hat{u}_{j|k}\right)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}+\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}.\end{aligned} \tag{A.12}$$

The term $\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)}$ is upper bounded as (Tuna et al. 2006)

$$\frac{\Upsilon_{k+N_c}(\Xi)}{\ell_c\left(\hat{x}_{k|k},\hat{u}_{k|k}\right)} \leq \prod_{i=1}^{N_c}\frac{L_i-1}{L_{i-1}} \leq (L-1)\left(\frac{L-1}{L}\right)^{N_c} \tag{A.13}$$

where L_i is the term of the sequence from assumption 4 and $L=\max\{L_i\}$. Then

$$
\begin{aligned}
\delta\omega = & \frac{\delta \Upsilon_{k+N_c}\left(\Xi\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \Delta_c^w, \\
\leq & \ \delta\left(L-1\right)\left(\tfrac{L-1}{L}\right)^{N_c} + \Delta_c^w.
\end{aligned}
\tag{A.14}
$$

If one choose the length of the control window with the following criterion

$$
N_c = \left\lceil \frac{\ln\left(\frac{\delta(L-1)}{1-\Delta_c^w}\right)}{\ln\left(\frac{L}{L-1}\right)} + 1 \right\rceil. \tag{A.15}
$$

the following inequality holds

$$
\delta\omega < 1. \tag{A.16}
$$

$\square$

B Derivation of $\dot{p}_t$

$$
\begin{aligned}
\dot{p}_t = & \frac{\Delta x}{|\Delta x|}\left(\dot{x}_t^{(1)} - \dot{x}_t^{(2)}\right), \\
= & \frac{\Delta x}{|\Delta x|}\left(a x_t^{(1)^3} + w_t^{(1)} + u_t^{(1)} - a x_t^{(2)^3} - w_t^{(2)} - u_t^{(2)}\right), \\
= & \frac{\Delta x}{|\Delta x|}\left(a\left(x_t^{(1)^3} - x_t^{(2)^3}\right) - K\Delta x + \Delta w_t\right), \\
= & \frac{\Delta x}{|\Delta x|}\left(a\Delta x\left(x_t^{(1)^2} + x_t^{(1)}x_t^{(2)} + x_t^{(2)^2}\right) - K\Delta x + \Delta w_t\right), \\
\leq & -K|\Delta x| + |\Delta x| a\left(x_t^{(1)^2} + x_t^{(1)}x_t^{(2)} + x_t^{(2)^2}\right) + |\Delta w_t|, \\
\leq & -K|\Delta x| + |\Delta x| a\frac{\left(x^{(1)^3} - x^{(2)^3}\right)}{\Delta x} + |\Delta w_t|, \\
\leq & -Kp_t + a\left(\left(y_t^{(1)} - v_t^{(1)}\right)^3 - \left(y_t^{(2)} - v_t^{(2)}\right)^3\right) + |\Delta w_t|, \\
\leq & -Kp_t + a|h^3\left(x_t^{(1)}\right) - h^3\left(x_t^{(2)}\right)| + |\Delta w_t|, \\
\leq & -Kp_t + ag|\Delta h_t| + |\Delta w_t|.
\end{aligned}
$$

$\square$