

# A multi-class structured dictionary learning method using discriminant atom selection

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## Abstract

In the last decade, traditional dictionary learning methods have been successfully applied to various pattern classification tasks. Although these methods produce sparse representations of signals which are robust against distortions and missing data, such representations quite often turn out to be unsuitable if the final objective is signal classification. In order to overcome, or at least to attenuate, such a weakness, several new methods which incorporate discriminant information into sparse-inducing models have emerged in recent years. In particular, methods for discriminant dictionary learning have shown to be more accurate than the traditional ones, which are only focused on minimizing the total representation error. In this work, we present both a novel multi-class discriminant measure and an innovative dictionary learning method. For a given dictionary, this new measure, which takes into account not only when a particular atom is used for representing signals coming from a certain class and the magnitude of its corresponding representation coefficient, but also the effect that such an atom has in the total representation error, is capable of efficiently quantifying the degree of discriminability of each one of the atoms. On the other hand, the new dictionary construction method yields dictionaries which are highly suitable for multi-class classification tasks. Our method was tested with two widely used databases for handwritten digit recognition and for object recognition, and compared with three state-of-the-art classification methods. The results show that our method significantly outperforms the other three achieving good recognition rates and additionally, reducing the computational cost of the classifier.

*Keywords:* Multi-class discriminant measure, Structured dictionary learning, Sparse coding, Handwritten digit recognition, Object recognition

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# 1. Introduction

Sparse representation of signals is considered a very powerful signal processing technique which has drawn massive interest in recent years mainly due to its success in solving a wide variety of problems in different fields such as biomedical signal processing [1, 2], computer vision [3] and image analysis [4], including image denoising [5], color image restoration [6] and image classification [7]. Roughly speaking, the problem of sparse representation consists of obtaining approximations of the involved signals in terms of linear combinations of only a few prescribed very simple characteristic signals taken from a large set [8, 9]. Besides providing a robust framework against distortions, missing data and noise, sparse representation of signals has many other advantages such as super resolution and dimensionality reduction [10].

A sparse representation problem (SRP) is usually divided into two sub-problems: an inference problem and a learning problem. The first one, which is often called “sparse coding”, consists of computing a representation vector satisfying a particular sparsity constraint given a predefined dictionary. The second one, which involves solving a more complex problem, consists of finding an “optimal”, in certain sense, dictionary for representing a given set of training signals. It is important to point out however, that most formulations of SRPs only focus on minimizing a prescribed total representation error and they do not take into account any a-priori discriminant information which could significantly improve the performance in the case of multi-object classification problems.

The first data-driven dictionary learning algorithms were originally developed almost two decades ago [8, 11, 12]. Some of them have their roots in probabilistic frameworks by considering the observed data as realizations of certain random variables [8, 11]. In [11] for example, the authors developed an algorithm for finding a redundant dictionary maximizing the likelihood function of the probability distribution of the data. In that work, an analytic expression for the likelihood function was derived by approximating the posterior distribution by Gaussian functions. On the other hand, an iterative approach for dictionary learning, known as the “Method for Optimal Directions” (MOD), was presented in [12]. The sparse coding stage of this method makes use of a greedy algorithm called “Orthogonal Matching Pursuit” (OMP) [13] followed by a simple dictionary updating rule.

A new iterative algorithm was proposed by Aharon *et al.* in [9]. This new approach, called “K Singular Value Decomposition” (KSVD), consists mainly of two stages: a sparse coding stage and a dictionary learning stage. The OMP algorithm is used in the sparse coding stage, which is followed by a dictionary updating step where the atoms are updated one at a time and the representation coefficients are allowed to change in order to minimize the total representation error.

In the last decade, the interest in developing algorithms based on sparse representation of signals for pattern recognition purposes has notably increased [7, 14, 15]. This is so because a large number of authors have proposed new supervised approaches for pattern recognition using sparse representations of signals. For instance, a discriminant version of the standard KSVD method applied to face recognition was presented by Zhang Q. *et al.* [7]. In that work, the authors included a discriminant term into the objective function of the standard KSVD algorithm. Results have shown that this modification constitutes an appropriate way to learn dictionaries which satisfy both criteria: low reconstruction error and high recognition rates. Also, Pham D. *et al.* [14] proposed an iterative method that simultaneously optimizes a dictionary and a linear classifier. The authors successfully used the method in an image categorization problem. More recently, a novel approach called “Label Consistent KSVD” (LC-KSVD) for dictionary learning was proposed in [15]. In that work a discriminant sparse representation and a single predictive linear classifier were efficiently integrated into the objective function.

However, besides supervised dictionary learning methods, many other new alternative options were presented [4, 16, 17]. These new alternatives are mainly based on the pursuit of discriminability of sparse representations through the development of “structured” or, more precisely, category-specific dictionary methods. In [4], a method for learning multiple dictionaries that uses the reconstruction errors yielded by these dictionaries on image patches to derive a pixel-wise classification. This algorithm has proved to be robust specially for local image classification tasks. A method for learning multiple non-redundant dictionaries for complex object categorization was proposed in [16]. This method was assessed on both visual object categorization and document classification image-related problems yielding competitive performances. In [17], a method that simultaneously optimizes both a structured dictionary (category-specific visual words for each feature) and a classifier was introduced. This method yielded good recognition rates showing a significant improvement over state-of-the-art object classification methods. A new method for structured dictionary learning was recently proposed by Sun *et al.* [18]. In that work, the learned dictionary was decomposed into class-specific sub-dictionaries for the classification that is conducted measuring the minimum reconstruction error among all the classes. The method was tested using both the synthetic data and the real-world data showing good performances.

In this work we propose a novel multi-class discriminant measure and a new dictionary learning method which yields structured dictionaries which are composed by category-specific sub-dictionaries specially constructed for multi-class classification purposes. Thus, the novelty of our approach is twofold. First, we introduce an

1 innovative and effective multi-class discriminant measure whose main property is pre-  
 2 cisely its capability for quantifying the discriminant degree of each one of the atoms  
 3 in a given dictionary. This measure takes into account not only when a particular  
 4 atom is used for representing a signal coming from a certain class and the magnitude  
 5 of its corresponding representation coefficient, but also the effect that such an atom  
 6 has in the total representation error. Secondly, this work presents a novel method  
 7 for discriminant structured dictionary learning which yields a dictionary increasing  
 8 the classifier recognition rate.

9 The organization of this article is as follows. A brief review of sparse representa-  
 10 tion of signals is presented in Section 2. In Section 3, we make a description of both  
 11 databases used in the experiments and we propose both a new discriminant measure  
 12 and a structured dictionary learning method. Section 4 details all the experiments,  
 13 while results and discussion are presented in Section 5. Finally, concluding comments  
 14 and future works are presented in Section 6.

## 15 2. Sparse representation of signals

Sparse representation is a signal processing technique that seeks the sparsest  
 representation of all the signals in a given set in terms of linear combinations of  
 certain basic waveforms. The sparse representation problem can be separated into  
 two sub-problems. Namely the so-called sparse coding problem and the dictionary  
 learning problem. We shall now proceed to describe in detail each one of these sub-  
 problems. For that, let  $\mathbf{x} \in \mathbb{R}^N$  be a discrete signal and let  $\Phi \in \mathbb{R}^{N \times M}$  (generally  
 with  $M \geq N$ ) be a dictionary whose columns  $\phi_j \in \mathbb{R}^N$  are atoms that we want to  
 use for obtaining representations of  $\mathbf{x}$  of the form  $\mathbf{x} = \Phi \mathbf{a}$ . Here, and in the sequel,  
 we shall refer to the vector  $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_M]^T \in \mathbb{R}^M$  as a “representation” of  
 $\mathbf{x}$ . Sparsity consists essentially of obtaining a representation with as few non-zero  
 elements as possible. A way of obtaining such representations consists of solving the  
 following problem:

$$(P_0) : \min_{\mathbf{a} \in \mathbb{R}^M} \|\mathbf{a}\|_0 \text{ subject to } \mathbf{x} = \Phi \mathbf{a},$$

16 where  $\|\mathbf{a}\|_0$  denotes the  $\ell_0$  pseudo-norm, defined as the number of non-zero elements  
 17 of  $\mathbf{a}$ . It turns out that imposing an exact representation of  $\mathbf{x}$  is a too restrictive  
 18 constraint, which makes  $(P_0)$  an NP hard problem [19, §1.8], yielding the approach  
 19 highly unsuitable for most practical applications.

Hence, the exact representation requirement  $\mathbf{x} = \Phi \mathbf{a}$  is often relaxed by allowing  
 small representation errors and imposing an upper bound on the  $\ell_0$  pseudo-norm of  
 the representations. Thus, a small error representation tolerant version of  $(P_0)$  is

defined as follows:

$$(P_0^q) : \min_{\mathbf{a} \in \mathbb{R}^M} \|\mathbf{x} - \Phi \mathbf{a}\|_2^2 \text{ subject to } \|\mathbf{a}\|_0 \leq q,$$

where  $q$  is a prescribed positive integer parameter. This formulation considers the presence of possible additive noise terms. In other words, it assumes that the signal  $\mathbf{x}$  can be represented in the form  $\mathbf{x} = \Phi \mathbf{a} + \mathbf{e}$ , where  $\mathbf{e} \in \mathbb{R}^N$  is a small energy noise term. Thus, this approach is more appropriate in a wide variety of real applications (such as biomedical signal and image processing) where the captured raw signals are always contaminated by noise. Several greedy strategies have been proposed for solving problem  $(P_0^q)$  [20, 13]. Among them, the OMP algorithm is perhaps the most commonly used strategy. This greedy algorithm ensures convergence to the projection of  $\mathbf{x}$  into the span of atoms in a given dictionary, in no more than  $q$  iterations. It is important to note that the representation vector  $\mathbf{a}$  has no more than  $q \ll M$  non-zero entries. Figure 1 shows an example of the representation vectors obtained with this  $(P_0^q)$  approach for two images of different classes coming from a widely used database which we shall describe in detail in Section 3. Note that most coefficients are strictly equal to zero.

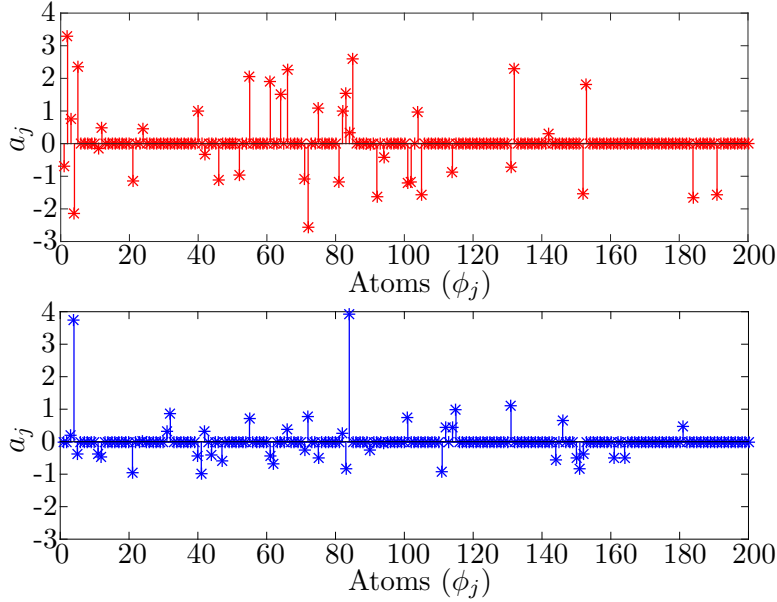


Figure 1: Example of two representation vectors of handwritten digits of two different classes obtained with the OMP algorithm.

Although pre-constructed dictionaries, such as the well known wavelet packets [21], typically lead to fast sparse coding, they are almost always highly restricted to certain classes of signals. Hence, due to their lack of generalization, new approaches introducing data-driven dictionary learning techniques have emerged. A dictionary learning problem associated to the data:  $q, M, N \in \mathbb{N}$ ,  $M \geq N$  and a collection of  $n$  signals in  $\mathbb{R}^N$ ,  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , can be formally written as:

$$(DL) : \min_{\substack{\Phi \in \mathbb{R}^{N \times M} \\ \mathbf{a}_i \in \mathbb{R}^M, \|\mathbf{a}_i\|_0 \leq q, 1 \leq i \leq n}} \sum_{i=1}^n \|\mathbf{x}_i - \Phi \mathbf{a}_i\|_2^2$$

1 The solution of this problem yields on one hand a dictionary  $\Phi$  and, on the other  
 2 hand, representations for all the signals in terms of that dictionary complying with  
 3 the sparsity constraint for each one of the “ $n$ ” involved signals  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . It is  
 4 important to point out that in such a process, the total representation error is min-  
 5 imized.

6 Although data-driven dictionary learning algorithms produce sparse representa-  
 7 tions of signals which are robust against distortions and missing data, such repre-  
 8 sentations quite often turn out to be unsatisfactory if the final objective is signal  
 9 classification. This is mainly due to the fact that those algorithms do not take  
 10 into account prior information concerning class membership. To overcome this flaw,  
 11 several alternative approaches producing sparse representations in terms of a unique  
 12 (and shallow) dictionary for signal classification were presented [7, 14, 15]. A different  
 13 approach is the construction of structured dictionaries composed by sub-dictionaries  
 14 whose atoms are discriminant, in certain sense, for each one of the classes, i.e. each  
 15 sub-dictionary has a group of atoms that are discriminant only for a particular class.  
 16 The use of structured dictionaries could be useful for reducing the features dimension,  
 17 avoiding over-fitting and optimizing the performance of a classifier, among others. In  
 18 recent years, there has been increasing interest in developing algorithms whose main  
 19 purpose is to obtain “optimal” sub-dictionaries to be used for signal classification  
 20 [1, 22, 23]. In [22], a method called “Clustering based Online Learning of Dictionar-  
 21 ies” (COLD) was presented. This algorithm makes use of the mean shift clustering  
 22 procedure [24] to identify modes in the distribution of the atoms and hence obtain a  
 23 dictionary of minimal size. Recently, Chen *et al.* [23] introduced a dictionary learn-  
 24 ing method for image and video editing tasks. In that work, the problem of seeking  
 25 an optimal dictionary is solved by using a symmetric version of the “Kullback-Leibler  
 26 Divergence” (KLD) [25]. This divergence has been successfully used for detecting  
 27 redundant atoms in a given dictionary. Our proposal consists of defining and using  
 28 a new discriminant measure for selecting the most discriminant atoms for each one  
 29 of the classes and use them for building a new structured dictionary.

### 3. Materials and methods

In this section we make a brief description of both databases used in the experiments. Additionally, we describe in detail both the new proposed multi-class discriminant measure and the novel structured dictionary learning method.

#### 3.1. Databases

One of the most popular databases used to assess Computer Vision and Pattern Recognition methods is the “Modified NIST” (MNIST) database [26]. This database has been widely used for assessing new methods including Deep Learning techniques [27], Extreme Learning Machines [28] and a many types of neural networks [29], among others. The MNIST database contains a total of 70,000 normalized and centered gray-scale images of handwritten digits ranging from 0 (zero) to 9 (nine), each one of size  $28 \times 28$  (leading to a feature vector of length 784). Also, the number of images per class varies from 5,421 to 6,742, corresponding to classes 5 and 1, respectively. Additionally, this database provides information about standard partitions used for training (60,000) and testing (10,000).

The CIFAR-10 database is comprised by 60,000 color images distributed into 10 different classes (objects) such as airplanes, cars, cats and horses, to name just a few, each one of size  $32 \times 32$  (leading to vectors of length 1024) [30]. This database is divided into two standard partitions for training and testing each one composed by 50,000 and 10,000 images, respectively. Also, it is known that the CIFAR-10 database is more challenging than MNIST since objects to be recognized have more variance with different poses and localizations.

Although each one of all original (raw) images coming from MNIST and CIFAR database can be represented as a single column vector, it becomes suitable to reduce its dimensionality for practical reasons. In this work, the image dimension reduction process is carried out by using the well known bi-cubic interpolation method [31] which is not only accurate, but also smooth and computationally efficient. This method was used on both databases for obtaining new (reduced) images each one of size  $16 \times 16$  (leading to vectors of length 256).

#### 3.2. A new discriminant measure

Discriminant dictionaries can be thought of as a collection of atoms specially learned for signal classification. These dictionaries not only produce accurate representations of the training signals (in terms of their waveforms) coming from different classes, but they also render their representations easy to distinguish by a suitable classifier. However, the problem of finding a discriminant dictionary is computationally very costly. A way to overcome the computational complexities entailed

1 by such a problem consists of defining an appropriate discriminant value functional  
 2 that independently evaluates each one of the atoms in a given dictionary. This sim-  
 3 plification is based on the assumption that each atom in the dictionary is used to  
 4 model specific characteristics that are not modeled by any one of the other atoms.  
 5 Thus, the discriminant information provided by a particular atom is different from  
 6 the information contributed by all the other atoms.

7 In a previous work [1], we presented a simple approach for quantifying the dis-  
 8 criminant degree of the atoms of a given dictionary  $\Phi$  in the context of a binary  
 9 classification problem. The approach essentially consists of counting the number  
 10 of times that a particular atom is used, i.e. it becomes “active” for representing  
 11 signals belonging to each one of both classes  $\ell = 1$  and  $\ell = 2$ . As a result of this  
 12 counting process, an activation frequency ( $\eta$ ) for each atom given the class, is con-  
 13 sidered. To quantify the discriminant degree of the  $j^{\text{th}}$ -atom ( $\phi_j$ , the  $j^{\text{th}}$ -column of  
 14  $\Phi$ ), the absolute difference of activation frequencies of that atom for classes  $\ell = 1$   
 15 and  $\ell = 2$  ( $|\eta_1^j - \eta_2^j|$ ) is computed. This value will be large if (and only if) the atom  
 16  $\phi_j$  is much more frequently used for representing signals in one of the two classes  
 17 and, in that case, it can be thought of as a quantifier of the capability of  $\phi_j$  to  
 18 supply important discriminant information regarding class membership. The use of  
 19 this discriminant quantifier gave rise to a method called Most Discriminative Col-  
 20 umn Selection (MDCS) for discriminant sub-dictionary construction [1]. The MDCS  
 21 method has shown to be robust for efficiently extracting meaningful features from  
 22 segments of pulse oximetry signals for detecting apnea-hypopnea events.

23 In this work we propose an extension of the measure described above to multi-  
 24 class classification problems. This extension consists of defining and using a new  
 25 multi-objective function aimed at quantifying the discriminant degree of each one of  
 26 the atoms in a given dictionary. This function will be defined as a convex combination  
 27 of three discriminant terms, all based on the affine sparse representations of the data.  
 28 In what follows, a detailed description of each one of such terms as well as a formal  
 29 definition of the function are presented.

### 30 *3.2.1. Activation frequency measure*

31 Conditional activation frequencies provide a reasonable starting point for deter-  
 32 mining the discriminant degree of individual atoms in a given dictionary. For this  
 33 reason, our approach begins by computing the activation frequency  $\eta_\ell^j$  of  $\phi_j$  given  
 34 the class  $\ell$ , for  $\ell = 1, 2, \dots, k$ . Moreover, the conditional activation probability of  
 35  $\phi_j$  given (that a signal  $\mathbf{x}$  belongs to) class  $\ell$  is defined as  $p_\ell^j \doteq P(a_j \neq 0 | \mathbf{x} \in \ell)$ .  
 36 Given a set of  $n_\ell$  signals belonging to class  $\ell$ , this conditional probability can be  
 37 approximated by the quotient  $\eta_\ell^j/n_\ell$ . Note that if the problem is balanced, i.e. if the



number of available signals belonging to each one of the  $k$  classes is the same, say  $\hat{n}$ , then  $\eta_\ell^j \propto p_\ell^j$ , more precisely  $\eta_\ell^j = k\hat{n}p_\ell^j$ , for all  $\ell$  and  $j$ . In this work, the problem of quantifying the discriminability of each atom is tackled by analyzing their individual contributions to the signal classification process. More specifically, a particular atom  $\phi_j$  is considered as having important discriminant information for class  $\ell$  signals if  $p_\ell^j > p_m^j$ , for all  $m \neq \ell$ . Hence, if  $\phi_j$  is discriminant for class  $\ell$ , the activation of the representation coefficient  $a_j$  will be strongly associated to class  $\ell$  membership. Since the performance of a classifier highly depends on the discriminability of their inputs, it is reasonable to think that using the representation coefficients  $a_1, a_2, \dots, a_M$  as inputs of a classifier, for atoms selected using that criterion, could result in good recognition rates.

For a given  $j$ ,  $1 \leq j \leq M$ , we shall denote by  $\ell_j^+$  the class that maximizes all conditional activation probabilities  $p_\ell^j$ , for all  $\ell = 1, 2, \dots, k$ , i.e. such that

$$p_{\ell_j^+}^j = \max_{1 \leq \ell \leq k} p_\ell^j. \quad (1)$$

In the (unlikely) case that there is more than one value of  $\ell$  maximizing  $p_\ell^j$ ,  $\ell_j^+$  is defined by randomly choosing one of them, for instance the smallest one (note that the order of the classes is completely irrelevant). Similarly, for a fixed  $j$ ,  $1 \leq j \leq M$ ,  $\ell_j^*$  is defined as the class leading to the second largest conditional activation probability, i.e. such that

$$p_{\ell_j^*}^j = \max_{\substack{1 \leq \ell \leq k \\ \ell \neq \ell_j^+}} p_\ell^j. \quad (2)$$

Here again if there is more than one value of  $\ell_j^*$  satisfying (2), then  $\ell_j^*$  is chosen randomly as any one of them.

Next we define the function  $m_{af} : \{1, 2, \dots, M\} \rightarrow \mathbb{R}_0^+$  by

$$m_{af}(j) \doteq \frac{p_{\ell_j^+}^j - p_{\ell_j^*}^j}{p_{\ell_j^+}^j}, \quad (3)$$

we shall refer to  $m_{af}(\cdot)$  as the ‘‘activation frequency measure’’.

Note that  $0 \leq m_{af}(\cdot) \leq 1$ . The atom  $\phi_j$  is said to be discriminant (for class  $\ell_j^+$ ) if and only if  $m_{af}(j) > 0$ . Clearly, within this setting, if an atom  $\phi_j$  is discriminant, it will be so only for the class  $\ell_j^+$ , otherwise it will be discriminant for none of them. Moreover, the value of  $m_{af}(j)$  can be thought of as a ‘‘measure’’ of the degree of discriminability of the atom  $\phi_j$  (for the corresponding class  $\ell_j^+$ ), based solely on the activation frequency information.

1 Figure 2 shows graphic representations of two examples of conditional activation  
 2 probabilities  $p_\ell^1$  and  $p_\ell^2$ , for  $\ell = 1, 2, \dots, 10$ , associated to atoms  $\phi_1$  (top) and  $\phi_2$   
 3 (bottom), respectively. The vertical bars represent the value of each conditional  
 4 activation probability  $p_\ell^1$  (top) and  $p_\ell^2$  (bottom), for  $\ell = 1, 2, \dots, 10$ . Clearly, for the  
 5 top case (atom  $\phi_1$ )  $\ell_1^+ = 4$  and  $\ell_1^* = 5$ ,  $m_{af}(1) > 0$  and therefore the atom  $\phi_1$  is  
 6 considered to be discriminant (for class 4). For the bottom case (atom  $\phi_2$ )  $\ell_2^+ = 2$   
 7 and  $\ell_2^* = 7$  (although these values could be interchanged), but since  $p_2^2 = p_7^2$ , one has  
 8  $m_{af}(2) = 0$  implying that  $\phi_2$  is not discriminant for class  $\ell_2^+$ , and therefore is not  
 discriminant for any one of the classes.

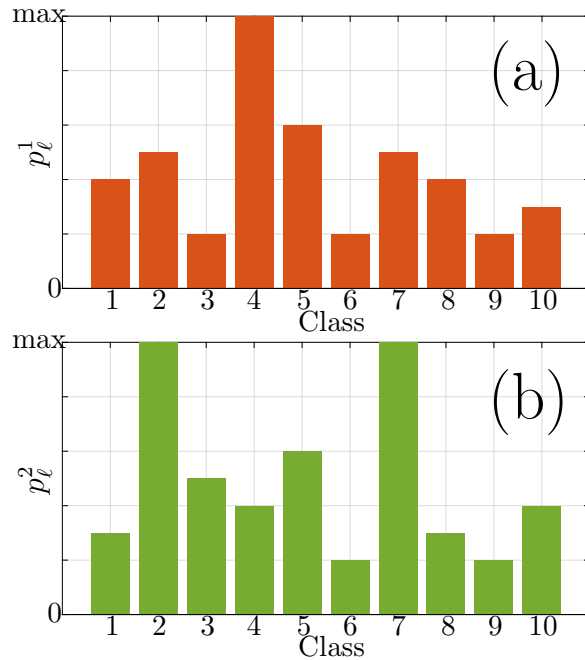


Figure 2: Vertical bars representing conditional activation probabilities for a class-specific discriminant atom  $\phi_1$  (top) and for a non-discriminant atom  $\phi_2$  (bottom) as proposed in this work.

9

10 *3.2.2. Coefficient magnitude measure*

On one hand, the sparse representation of signals provides valuable information regarding the activation of atoms and, on the other hand, it can highlight important characteristics or features contained in particular event related waveforms of signals or images such as brightness variations in images and slight changes in biomedical signals, to name but a few. With the above observation in mind, we proceed now to define a second measure that takes into account the magnitude of the representation coefficients. For that, given an atom  $\phi_j$ , let  $\ell_j^+$  and  $\ell_j^*$  be the classes as defined

in (1) and (2), respectively, and let  $\mathbf{A}_{\ell_j^+}$  and  $\mathbf{A}_{\ell_j^*}$  the matrices which provide the sparse representations of  $\mathbf{X}_{\ell_j^+}$  and  $\mathbf{X}_{\ell_j^*}$ , respectively, in terms of the dictionary  $\Phi$ , i.e.  $\mathbf{X}_{\ell_j^+} = \Phi \mathbf{A}_{\ell_j^+}$  and  $\mathbf{X}_{\ell_j^*} = \Phi \mathbf{A}_{\ell_j^*}$ . Additionally, let  $q_{\ell_j^+}^j$  denote the quotient  $\|[\mathbf{A}_{\ell_j^+}]_{j,:}\|_1/n_{\ell_j^+}$ , where  $[\mathbf{A}_{\ell_j^+}]_{j,:}$  represents the  $j^{\text{th}}$ -row of the matrix  $\mathbf{A}_{\ell_j^+}$ . The coefficient magnitude measure is the function  $m_{cm} : \{1, 2, \dots, M\} \rightarrow \mathbb{R}_0^+$  defined by

$$m_{cm}(j) \doteq \frac{q_{\ell_j^+}^j - q_{\ell_j^*}^j}{q_{\ell_j^+}^j}. \quad (4)$$

Here again  $0 \leq m_{cm}(\cdot) \leq 1$ . Based on this measure, an atom  $\phi_j$  is said to be discriminant (for the class  $\ell_j^+$ ) if and only if  $m_{cm}(j) > 0$  and, in that case, the value of  $m_{cm}(j)$  quantifies the corresponding degree of discriminability of  $\phi_j$  for the class  $\ell_j^+$ .

### 3.2.3. Representation error measure

We now proceed to describe the third measure for quantifying the discriminant degree of each atom in a dictionary. This measure takes into account the contribution of each atom  $\phi_j$  to the total representation error. Let  $\mathbf{A}_{\ell} \doteq [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_{n_{\ell}}]$  be the matrix providing the sparse representation of  $\mathbf{X}_{\ell} \doteq [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{n_{\ell}}]$ , as in the previous measure. Clearly, the contribution of the class  $\ell$  to the total representation error can be written as [9]

$$\begin{aligned} \sum_{i=1}^{n_{\ell}} \|\mathbf{x}_i - \Phi \mathbf{a}_i\|_2^2 &= \|\mathbf{X}_{\ell} - \Phi \mathbf{A}_{\ell}\|_F^2 \\ &= \left\| \mathbf{X}_{\ell} - \sum_{j=1}^M \phi_j [\mathbf{A}_{\ell}]_{j,:} \right\|_F^2 \\ &= \left\| \left( \mathbf{X}_{\ell} - \sum_{i \neq j} \phi_i [\mathbf{A}_{\ell}]_{i,:} \right) - \phi_j [\mathbf{A}_{\ell}]_{j,:} \right\|_F^2 \\ &\doteq \left\| \mathbf{E}_{\ell}^j - \phi_j [\mathbf{A}_{\ell}]_{j,:} \right\|_F^2, \end{aligned} \quad (5)$$

where  $\mathbf{E}_{\ell}^j$  denotes the total representation error for all class  $\ell$  signals when  $\phi_j$  is removed. Hence, a large value of  $\mathbf{E}_{\ell}^j$  indicates that the contribution of  $\phi_j$  to the representation of class  $\ell$  signals is large. We then define a “representation error

measure”  $m_{re} : \{1, 2, \dots, M\} \rightarrow \mathbb{R}_0^+$  by

$$m_{re}(j) \doteq \frac{r_{\ell_j^+}^j - r_{\ell_j^*}^j}{r_{\ell_j^+}^j}, \quad (6)$$

1 where  $r_{\ell}^j \doteq \mathbf{E}_{\ell}^j/n_{\ell}$ , for  $\ell = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, M$ .

2 Here again  $0 \leq m_{re}(\cdot) \leq 1$ , and an atom  $\phi_j$  is said to be discriminant (for class  
3  $\ell_j^+$ ) with respect to this measure if and only if  $m_{re}(j) > 0$ . In such a case, the value  
4 of  $m_{re}(j)$  quantifies the corresponding degree of discriminability.

#### 5 3.2.4. A combined discriminant measure

6 Each one of the three previously defined measures takes into account different  
7 properties related to the discriminability of each one of the atoms (in a given dic-  
8 tionary). It is then reasonable to think of a measure that appropriately combines  
9 all three of them. With that in mind, given two positive parameters  $\alpha$  and  $\beta$ , with  
10  $\alpha + \beta \leq 1$ , we define the function  $m_{\alpha,\beta} : \{1, 2, \dots, M\} \rightarrow \mathbb{R}_0^+$  as

$$m_{\alpha,\beta}(j) \doteq \alpha m_{af}(j) + \beta m_{cm}(j) + (1 - \alpha - \beta) m_{re}(j). \quad (7)$$

11 We shall refer to  $m_{\alpha,\beta}$  as the “combined discriminant measure”. Clearly, as  $\alpha$   
12 and  $\beta$  vary between 0 and 1, (7) exhausts all possible convex combinations of the  
13 three single measures  $m_{af}$ ,  $m_{cm}$  and  $m_{re}$ . A challenging problem, on which we shall  
14 shed some light in Section 4.3, consists precisely of finding the “optimal” pair of  
15 parameters  $(\alpha^*, \beta^*)$  leading to the best recognition rate, for a given problem.

#### 16 3.3. Dictionary learning algorithm

17 Supervised dictionary learning methods have observed great interest in recent  
18 years. Implementations of these methods were originally focused on efficiently learn-  
19 ing simple dictionaries (unstructured) that incorporate information of “discriminabil-  
20 ity” (in terms of signal classification) in their optimization process. This information  
21 can be introduced to the learning model by considering different discriminant crite-  
22 ria [32, 33, 34]. The most commonly used criteria are the so called “softmax” cost  
23 function [16], Fisher criterion [35] and linear predictive classification error [7, 14], to  
24 name just a few.

25 Although there exist several ways to simultaneously optimize both a dictionary,  
26 i.e. to solve a representation learning problem, and a classifier, i.e. to find a solution  
27 to a classification problem, a very often used strategy consists simply of dividing  
28 that problem into two sub-problems [4, 16]. Hence, it is possible to use all existing

traditional dictionary learning techniques, such as MOD and KSVD, and therefore train a single classifier at a later stage. Our proposal is based precisely on this strategy but introducing class information in the dictionary learning stage. For that, we propose a new method for multi-class structured dictionary learning called “Discriminant Atom Selection KSVD” (DAS-KSVD) in which we use the proposed discriminant measure  $m_{\alpha,\beta}$  to efficiently select class-specific discriminant atoms from some given “auxiliary” dictionaries to iteratively construct a structured one. The DAS-KSVD method aims at building a structures dictionary  $\Phi_D^{(I)}$  by stacking side-by-side  $k$  sub-dictionaries  $\Phi_\ell$ , each one of size  $N \times I$ , for all  $\ell = 1, 2, \dots, k$ ,  $\Phi_D^{(I)} \doteq [\Phi_1 \Phi_2 \dots \Phi_k]$ . It is  $N \times n$  signal matrix  $\mathbf{X}_{trn} \doteq [\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_k]$ , important to point out that each sub-dictionary  $\Phi_\ell$  contains atoms that are discriminant, in terms of  $m_{\alpha,\beta}$ , for class  $\ell$  signals.

We now proceed to describe the building steps of the proposed DAS-KSVD method in more detail (Algorithm 1). Here, and in the sequel, we shall consider the vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  as realizations of a particular  $N$ -dimensional random vector  $\mathcal{X}$ . Given an  $N \times n$  signal matrix  $\mathbf{X}_{trn} \doteq [\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_k]$ , composed of  $n = \sum_{\ell=1}^k n_\ell$  samples, the required sparsity level  $q$ , the redundancy factor  $r_f$ , the number  $t$  of class  $\ell$  training signals,  $t \ll n_\ell$ , the number of iterations  $I$  and the class label vector  $\mathbf{c}$ , the proposed algorithm begins by assigning an initial uniform probability distribution  $p_0$  over  $\mathbf{X}_{trn}$  so  $p_0(i) = 1/n$ , for all  $i$  (Alg. 1, line 2). The value of  $p_0(i)$  is the probability that a training signal  $\mathbf{x}_i$  is selected from  $\mathbf{X}_{trn}$  in order to construct a new sampled “learning” matrix  $\mathbf{X}_{lrn}$  that is used specifically for learning the initial dictionary  $\Phi$ . Additionally, if a certain training signal  $\mathbf{x}_i$  is used for learning the dictionary  $\Phi$  in a particular iteration, then it is desirable that such a signal be less likely than the other ones in the following iterations. Hence, promoting diversity in this way one might think that the final learned atoms are capable of highlighting different intrinsic properties of the training data.

The iterative process of this algorithm (Alg. 1, lines from 3 to 10) begins by statistically sampling  $t$  samples (note that  $t \ll n_\ell$ , for instance 10 times smaller) from each class  $\ell$  signal matrix  $\mathbf{X}_\ell$ . As a result of such a sampling process, a matrix  $\mathbf{X}_{lrn}$  of size  $N \times (t * k)$  is built (Alg. 1, line 4). Also, to compute the distribution  $p_{l+1}$  from both  $p_l$  and  $\mathbf{X}_{lrn}$ , we multiply the value of  $p_l(i)$  by a non-negative number  $\tau_1 < 1$  if (and only if)  $\mathbf{x}_i$  has been selected, i.e.  $p_{l+1}(i) = p_l(i)\tau_1$  (in that case  $p_{l+1}(i) < p_l(i)$ ). Otherwise  $p_l(i)$  is left unchanged. It is important to point out that an appropriate normalization of these weights forcing them to sum one is needed. Figure 3 shows graphic representations of five probability distributions  $p_l$ , for  $l = \{1, 5, 10, 15, 20\}$ . It can be observed that, at the first iteration, all samples have the same probability to be selected. In addition, see that the probability value of most samples decreases

1 as the iteration order increases.

2 In order to increase robustness, all training signals used to learn the dictionary  $\Phi$   
 3 (Alg. 1, line 5) are also degraded by incorporating an additive zero-mean Gaussian  
 4 noise  $\epsilon_{l,i}$  whose magnitude increases proportionally according to the iteration level.  
 5 The magnitude of the noise is updated by  $\epsilon_{l,i} = l\sigma_i\tau_2$ , where  $\sigma_i$  is the variance of  $\mathbf{x}_i$   
 6 and  $\tau_2$  is a (prescribed) non-negative number,  $\tau_2 < 1$ . For instance, the magnitude  
 7 of the noise associated to the signal  $\mathbf{x}_1$  at iteration 5 will be  $\epsilon_{5,1} = 5\sigma_1\tau_2$ . It is  
 8 important to point out however that, the first iteration ( $l = 1$ ) of the proposed  
 9 learning algorithm leaves the original image undegraded. On the other hand, the  
 10 dictionary  $\Phi$  is learned by means of the traditional unsupervised KSVD algorithm  
 11 [9]. Then the sparse matrix  $\mathbf{A}_{lrn}$  is obtained by applying the previously mentioned  
 12 OMP algorithm (Alg. 1, line 6). The reason for having chosen this pursuit algorithm  
 13 is because it guarantees convergence to the projection of each one of the signals into  
 14 the span of the dictionary atoms, in no more than  $q$  iterations leaving the rest of the  
 15 coefficients equal to zero.

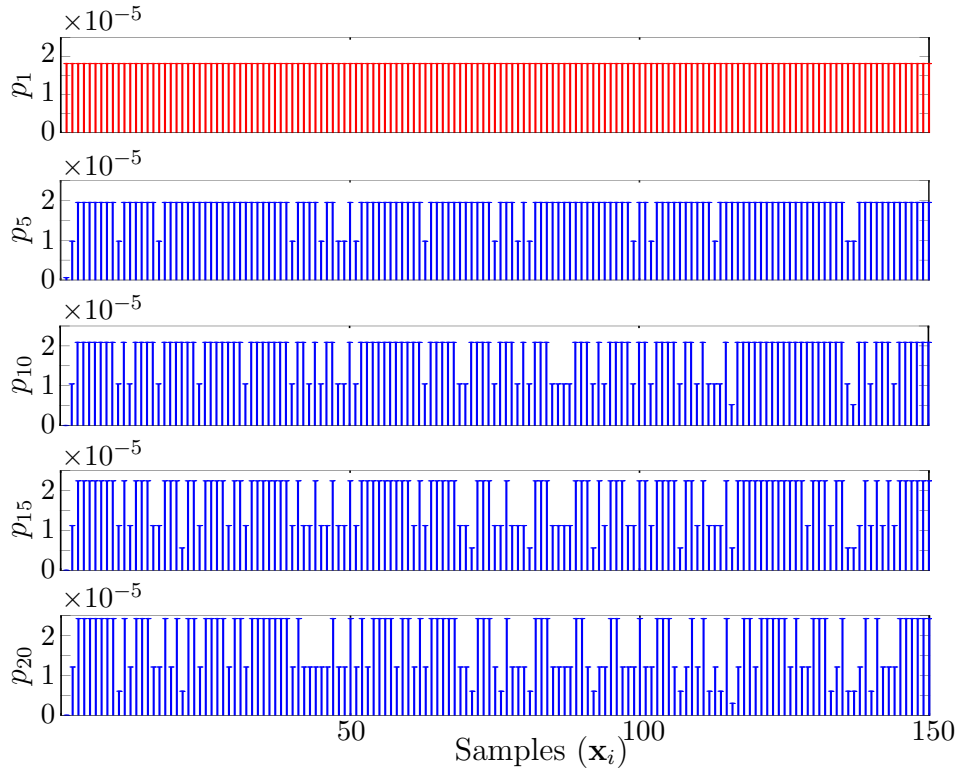


Figure 3: Data probability distributions for five different iterations of the proposed algorithm.

As previously mentioned, at the beginning of each iteration, the standard unsupervised KSVD algorithm was used to learn a dictionary  $\Phi$  of size  $256 \times 256$ . Note that this dictionary learning stage does not take into account any information concerning class membership. Additionally, the sampled subset of  $t * k$  signals used to learn the dictionary was appropriately degraded by incorporating additive Gaussian noise with different magnitudes. Left and right sides of Figure 4 show examples of atoms coming from the dictionary  $\Phi$  that were learned at iterations 1 and 20, respectively. It can be seen that, at the first iteration, the dictionary is learned by means of noise-free input signals. On the other hand, the dictionary learned at iteration 20 still preserves the structure of the handwritten digits on a blurred background.

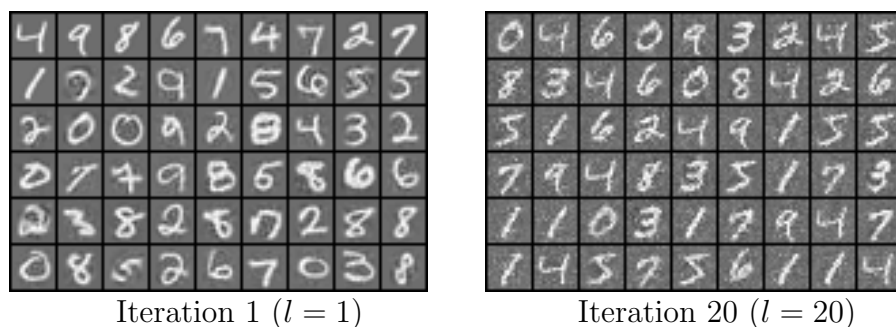


Figure 4: Some atoms of the dictionary  $\Phi$  for two different iterations of the DAS-KSVD algorithm. Iteration 1 (left) and iteration 20 (right). Images taken from the MNIST database.

The proposed discriminant approach consists of optimizing and using the new combined discriminant measure  $m_{\alpha,\beta}$  for selecting the most discriminant atoms of  $\Phi$  for each one of the  $k$  classes (Alg. 1, lines from 7 to 9). As explained in Section 3.2, the value of  $m_{\alpha,\beta}(j)$  corresponds to the degree of discriminability of the atom  $\phi_j$  for one (and only one) class, which is denoted by  $\ell_j^+$ . Note that the process of selecting the most discriminant atoms carries a serious trouble since the problem of finding the optimal pair of parameters  $(\alpha^*, \beta^*)$  is very challenging. For more details about the tuning of that pair of parameters, we refer the reader to Section 4.3 and Appendix A. Also, the construction of the sub-dictionary  $\Phi_d$  (Alg 1, line 8) basically consists of taking one-by-one the most discriminant atoms of  $\Phi$  for each one of the  $k$  classes and stacking them side-by-side. In the case that there is more than one  $\ell_j^+$  class-related candidate complying with the proposed discriminant criterion,  $\phi_j$  is defined as the atom that maximizes all possible values of  $m_{\alpha^*,\beta^*}$ . Otherwise, in case that  $\Phi$  lacks of discriminant atoms, the signal selection process (Alg. 1, line 4) is restarted.

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**Algorithm 1** Pseudocode of the new DAS-KSVD method
 

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1: procedure DAS-KSVD( $\mathbf{X}_{trn}, q, r_f, t, I, \mathbf{c}$ )
2:    $p_0(i) = 1/n$ , for all  $i$ 
3:   for  $l \leftarrow 0, I - 1$  do
4:      $[\mathbf{X}_{lrn}, p_{l+1}] \leftarrow \text{SAMPLEDATA}(\mathbf{X}_{trn}, t, p_l, l)$ 
5:      $\Phi \leftarrow \text{KSVD}(\mathbf{X}_{lrn}, r_f, q)$ 
6:      $\mathbf{A}_{lrn} \leftarrow \text{OMP}(\mathbf{X}_{lrn}, \Phi, q)$ 
7:      $m_{\alpha^*, \beta^*} \leftarrow \text{DISCMEASURE}(\mathbf{A}_{lrn}, \mathbf{c}, q)$ 
8:      $\Phi_d \leftarrow \text{GETATOMS}(\Phi, m_{\alpha^*, \beta^*})$ 
9:      $\Phi_D^{(i)} \leftarrow \text{SAVEATOMS}(\Phi_d)$ 
10:  end for
11:  return  $\Phi_D^{(I)}$ 
12: end procedure

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### 1 3.4. Classifier

In this work a Multilayer Perceptron (MLP) neural network is used in order to assess the proposed method. The MLP neural network is one of the most popular classes of neural networks whose architecture consists of a fully connected assembly of single artificial neurons. The MLP neural network is typically comprised by an input layer, one (or more) hidden layers and an output layer [36]. The inputs (features) are processed layer-by-layer moving forward through the network. Each artificial neuron receives one (or more) inputs from its preceding nodes, processes the information and produces an output that is transmitted to the next node. The output of each neuron is reached by applying an activation (transfer) function (linear or not) to the weighted sum of the inputs plus a bias term. More precisely, the output of a neuron  $y_j$  is defined as

$$y_j = f \left( \sum_{i=1}^d \omega_{ji} x_i + \omega_{j0} \right) = f \left( \sum_{i=0}^d \omega_{ji} x_i \right), \quad (8)$$

2 where the transfer function is denoted by  $f(\cdot)$ , and the weights that connect the  
 3  $i^{\text{th}}$ -input to the  $j^{\text{th}}$ -neuron for a given layer is represented by  $\omega_{ji}$ .

4 Since the MLP neural network training process is supervised, the desired outputs  
 5 (labels) are required. The most popularly used method for training MLP neural  
 6 networks is the back-propagation algorithm [37]. This algorithm iteratively adjusts  
 7 the synaptic weights in the network by minimizing a given measure which quantifies  
 8 the difference between the current output vector and the desired one.



## 4. Experiments

In this section we present a brief description regarding the experimental setup. Additionally, we make a brief recall of the evaluation metric used for assessing the proposed dictionary learning method. Finally, we comment on appropriate ways for tuning the parameters.

### 4.1. Experimental setup

Two different databases, namely, the MNIST database and the CIFAR-10 database are used to evaluate the proposed DAS-KSVD method. Two types of experiments are designed. First, an exhaustive set of simulation experiments are carried out to demonstrate the effectiveness and robustness of the proposed method for handwritten digit recognition using the well-known MNIST database. On the other hand, the second set of experiments are conducted to further assess the proposed method for object recognition on the CIFAR-10 database.

As mentioned above, the proposed DAS-KSVD method is first evaluated using standard partitions for training and testing of the MNIST database. Although it is not a strict requirement, our experiment is performed using a balanced set of training and validation samples. For that, subsets consisting of 4,000 and 1,000 images for each one of the classes coming from the standard partition of the training database are randomly chosen. Hence, new training and validation matrices ( $\mathbf{X}_{trn}$  and  $\mathbf{X}_{val}$ ) comprised by 40,000 and 10,000 samples, respectively, are built. It is important to point out however that, the standard partition of the testing database  $\mathbf{X}_{tst}$  of size  $256 \times 10,000$  is left unchanged. It becomes appropriate to mention that the matrix  $\mathbf{X}_{trn}$  is used both for dictionary learning and training the MLP neural network while the matrix  $\mathbf{X}_{val}$  was used for testing the MLP neural network as well as for parameters tuning. Furthermore, the matrix  $\mathbf{X}_{tst}$  is only considered to perform the final test.

We shall now proceed to describe the parameter settings for DAS-KSVD that are used in the first experiment. We evaluated the effect that produces the size of  $\Phi_D^{(I)}$  in the final recognition rate. For that, we have considered four structured dictionaries denoted by  $\Phi_D^{(5)}$ ,  $\Phi_D^{(10)}$ ,  $\Phi_D^{(15)}$  and  $\Phi_D^{(20)}$  which are composed by 50, 100, 150 and 200 atoms, respectively. Hence, the DAS-KSVD algorithm is set to execute 20 iterations, i.e.  $I = 20$ .

### 4.2. Evaluation metric

Overall accuracy rate constitutes one of the most popular performance measures used to assess pattern recognition-related methods. The accuracy measure (Acc) is defined as the proportion of correctly predicted testing samples. Let  $n$  the number of testing samples,  $\lambda_i$  and  $\hat{\lambda}_i$  the label and prediction, respectively, regarding  $\mathbf{x}_i$  and

$\delta(x, y)$  the well known delta function whose output is true (one) if  $x = y$  and false (zero) otherwise. The Acc measure is defined as:

$$\text{Acc} = \frac{1}{n} \sum_{i=1}^n \delta(\lambda_i, \hat{\lambda}_i). \quad (9)$$

### 1 4.3. Parameters tuning

2 Although the pursuit for discriminant atoms is perhaps one of the most challeng-  
 3 ing issues to be addressed in this work, finding optimal pair of parameters  $(\alpha^*, \beta^*)$   
 4 leading to the best recognition rate is also a very difficult task. However, the problem  
 5 of finding that optimal pair of parameters strongly depends on the application under  
 6 study. For that reason, we propose applying the well known and widely used “grid  
 7 search” method for parameter optimization. For more details regarding grid search  
 8 method, we refer the reader to Appendix A. In what follows, the final choice of the  
 9 remaining parameters of the proposed algorithm are described.

10 At each iteration of the proposed DAS-KSVD method, one (and only one) dis-  
 11 criminant atom for each one of the  $k$  classes is selected. Hence, each iteration of this  
 12 method generates  $k$  discriminant atoms and therefore, if the algorithm is configured  
 13 to perform  $I$  iterations, then the final structured dictionary will be composed by  
 14  $I * k$  discriminant atoms. In order to explore the effect of the final structured dic-  
 15 tionary size, the experiments were performed by considering a total of 20 iterations,  
 16 i.e.  $I = 20$ . Thus, the final discriminant dictionary  $\Phi_D^{(I)}$  is composed by 200 atoms  
 17 (assuming  $k = 10$ ). On the other hand, the number of samples for each class used  
 18 to learn the full dictionary was set to  $t = 500$ .

19 As described in Section 3.3,  $\tau_1$  and  $\tau_2$  are two parameters ( $0 \leq \tau_1, \tau_2 < 1$ ) that  
 20 need to be adjusted and fixed. Several trials were performed in order to obtain  
 21 appropriate values for those parameters. A value of  $\tau_1 = 0.5$  was finally selected  
 22 and used in our experiments. Additionally, it was found that a value of  $\tau_2 = 0.1$   
 23 presented the best trade-off between image degradation and iteration order.

24 The standard KSVD algorithm starts by performing a random selection of 256  
 25 samples coming from the learning signal matrix  $\mathbf{X}_{lrm}$ . Note that the redundancy  
 26 factor ( $r_f$ ) used for constructing the dictionary is equal to one, i.e.  $M = N =$   
 27 256. Also, the maximum number of KSVD iterations was fixed to 50 in the code.  
 28 It is also well known that the KSVD algorithm internally computes sparse codes  
 29 representing each one of all involved signals. These codes were obtained by means of  
 30 OMP algorithm. To establish an appropriate sparsity level, a great variety of sparse  
 31 solutions were tested. It was found that a sparsity degree of 20% presents the best  
 32 trade-off between discriminability and representativity of all signals.

The MLP neural network training process was performed using back-propagation method. This algorithm was optimized minimizing the Mean Squared Error (MSE) function through Scaled Conjugate Gradient (SCG) method. Also, the output of each neuron was determined by applying a saturating linear transfer function. Additionally, the structure of the MLP neural network was configured such that the sizes of its hidden and input layers are equal.

## 5. Results and discussion

### 5.1. Evaluation on the MNIST database

As already explained above, the matrices denoted by  $\mathbf{A}_{trn}$  and  $\mathbf{A}_{val}$  provide the sparse representations of  $\mathbf{X}_{trn}$  and  $\mathbf{X}_{val}$ , respectively, in terms of a dictionary  $\Phi$  through  $\mathbf{X}_{trn} = \Phi\mathbf{A}_{trn}$  and  $\mathbf{X}_{val} = \Phi\mathbf{A}_{val}$ . Also, the feature vectors  $\mathbf{a}_i$  comprising the matrices  $\mathbf{A}_{trn}$  and  $\mathbf{A}_{val}$  were used as inputs for training and testing, respectively, the MLP neural network. The final test was performed by taken into account the standard partition of the testing dataset  $\mathbf{X}_{tst}$  and each one of the previously learned structured dictionaries  $\Phi_D^{(I)}$ . The matrix  $\mathbf{A}_{tst}$  was obtained by means of the OMP algorithm. Also, the inputs of the already trained MLP neural networks are the feature vectors  $\mathbf{a}_i$  coming from  $\mathbf{A}_{tst}$  and, moreover, the outputs of these networks are evaluated to compute the final accuracy. In addition, structured dictionaries composed by 50, 100, 150 and 200 discriminant atoms were evaluated. Mean and standard deviation of the classification results over 10 rounds were found to be 94.87% ( $\pm 0.33\%$ ), 94.79% ( $\pm 0.30\%$ ), 94.36% ( $\pm 0.27\%$ ) and 91.25% ( $\pm 0.63\%$ ) for feature vector sizes of 200, 150, 100 and 50, respectively. Also, Table 1 presents a comparative summary of the best recognition rates yielded by MLP neural networks trained using as input the matrix  $\mathbf{A}_{trn}$  obtained by taken into account each one of the evaluated structured dictionaries. Also, details regarding the required number of weights of the MLP neural network for each one of such dictionaries are included. It is important to point out that these results were obtained by considering a fixed hidden layer size coinciding with the input feature vector size. Maximal accuracy rates of 96.2, 95.9, 95.0 and 92.2 were obtained for feature vector sizes of 200, 150, 100 and 50, respectively. Hence, results show that “discriminant” feature vectors of length 200 are the best option for handwritten digits recognition. On the other hand, the last column of Table 1 shows the total number of weights required to train each one of the MLP neural networks.

Lecun *et al.* [29] tested several configurations of one-hidden layer fully connected MLP neural networks trained for handwritten digit recognition. One of them consists of directly using the original (raw) data, i.e. without tacking into account any signal

Table 1: Best recognition rates on the test set yielded by MLP neural networks using DAS-KSVD feature vectors as well as the number of weights required for their training.

Dictionary	Classifier	Acc (%)	Number of weights
$\Phi_D^{(5)}$	MLP-50-50-10	92.23	3,060
$\Phi_D^{(10)}$	MLP-100-100-10	95.03	11,110
$\Phi_D^{(15)}$	MLP-150-150-10	95.90	24,160
$\Phi_D^{(20)}$	MLP-200-200-10	<b>96.20</b>	42,210

pre-processing or feature selection, as input of the classifier. Thus, vectors containing 784 features corresponding to images of size  $28 \times 28$  were used as inputs of the classifier. The first two rows of Table 2 shows maximal percentages of accuracy rates (Acc) yielded by MLP neural networks with 300 (MLP-784-300-10) and 1000 (MLP-784-1000-10) neurons in their hidden layer. The number of training weights for each one of the networks are also included in the last column. Accuracy rates on the standard test partition of 95.3% and 95.5% were yielded by MLP neural networks with 300 and 1000 hidden neurons, respectively. It can be observed that, as a result of increasing the number of hidden neurons (from 300 to 1000), a slight improvement in the result was achieved. Also, the number of weights of the network has increased from 238,510 to 795,010, which represent an increment of 333%.

Table 2 also shows a comparative summary of the results yielded by MLP neural networks with a reduction in the dimension of the feature vectors. For that, the proposed DAS-KSVD method was used for obtaining feature vectors of length 200. As shown in Table 1, structured dictionaries composed by 200 discriminant atoms ( $\Phi_D^{(20)}$ ) are the best option for handwritten digit recognition. Clearly, the use of small dimensional feature vectors produce a significant dimension reduction but retaining discriminant information and therefore, the computing time required for classification is reduced. Thus, the number of input units of the MLP neural network was reduced (from 784 to 200) in 74.49% compared with those required by the original raw data. The table shows the average over 10 rounds of accuracy rates yielded by MLP neural networks with 200 input units while varying the number of hidden neurons from 50 to 1000. The last column of this table shows the required number of training weights. Accuracy rates on the standard testing dataset of 96.4% and 96.7% were achieved by MLP neural networks with 300 (MLP-200-300-10) and 1000

Table 2: Test recognition accuracy on MNIST database. A comparative summary of classification performance of different methods with various MLP configurations.

Method	Classifier	Acc (%)	Number of weights
Raw data [29]	MLP-784-300-10	<b>95.3</b>	<b>238,510</b>
	MLP-784-1000-10	95.5	795,010
DAS-KSVD	MLP-200-50-10	<b>95.3</b>	<b>10,560</b>
	MLP-200-100-10	96.1	21,110
	MLP-200-200-10	96.2	42,210
	MLP-200-300-10	96.4	63,310
	MLP-200-1000-10	<b>96.7</b>	<b>211,010</b>
KSVD [9]	MLP-200-50-10	93.5	10,560
	MLP-200-100-10	92.8	21,110
	MLP-200-200-10	92.3	42,210
	MLP-200-300-10	92.8	63,310
	MLP-200-1000-10	92.7	211,010
LC-KSVD2 [15]	MLP-200-50-10	91.8	10,560
	MLP-200-100-10	91.9	21,110
	MLP-200-200-10	92.0	42,210
	MLP-200-300-10	92.1	63,310
	MLP-200-1000-10	92.3	211,010

(MLP-200-1000-10) hidden neurons, respectively. Additionally, the performance of MLP neural networks with 50, 100 and 200 hidden neurons were tested without showing significant improvements in the results.

It is also important to point out that the classifier MLP-200-50-10 (DAS-KSVD method) has achieved the same recognition rate (95.3%) as MLP-784-300-10 (Raw data) using a MLP neural network composed by only a 4.42% of the required weights. It was also found that taking into account the best option that uses the original raw data as inputs of the classifier (MLP-784-1000-10), it has 795,010 training weights while DAS-KSVD method (MLP-200-1000-10) has not only 211,010 weights, but also increases a 1.2% in the performance of the classifier. As a result of that analysis, one might think that the proposed DAS-KSVD method produces a significant dimension reduction while enhancing the overall recognition rate. Summing up, it was demonstrated that using the proposed DAS-KSVD method for dimension reduction undoubtedly enhances the recognition rate of MLP neural networks.

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1 We have compared the performance of the new DAS-KSVD method with the  
2 standard KSVD method as well as with the discriminant-based LC-KSVD2 method.  
3 It can be observed from Table 2 that the proposed DAS-KSVD method outperform all  
4 the others showing robustness and effectiveness with the same size of the dictionary  
5 in the recognition of handwritten digits images coming from MNIST database. The  
6 maximum recognition rate yielded by the DAS-KSVD method was 96.7% which  
7 clearly outperforms those yielded by both KSVD (93.5%) and LC-KSVD2 (92.3%)  
8 methods.

9 We have also evaluated the statistical significance of the results presented in  
10 Table 2 by computing the probability that the DAS-KSVD method yields better  
11 recognition rates than all the other evaluated methods ( $P(\epsilon_{ref} < \epsilon)$ ). In order to  
12 perform this test we assumed the statistical independence of the classification errors  
13 for each image and we approached the error's Binomial distribution by means of a  
14 Gaussian distribution. This is possible because we have a sufficiently high number  
15 of testing samples (10,000). In this way, for 95.5% and 96.7% corresponding to  
16 recognition rates yielded by "Raw data" that produced the best performance among  
17 all methods considered in the experiments and the new proposed one (DAS-KSVD),  
18 respectively, we have that  $P(\epsilon_{ref} < \epsilon) > 0.9999$ .

### 19 5.2. Evaluation on the CIFAR-10 database

20 We also assessed the proposed DAS-KSVD method on the CIFAR-10 database  
21 to further visualize its effectiveness on another task. As previously mentioned, the  
22 CIFAR-10 database contains 60,000 color images separated into 10 different objects,  
23 each one of size  $32 \times 32$  (leading to vectors of length 1024). To carry out the second  
24 set of experiments, the images were converted to gray-scale and also resized to  $16 \times 16$   
25 (leading to vectors of length 256). Indeed, it is of our interest to study the effect that  
26 produces the dimension of the data, i.e. the dimension of the underlying dictionary  
27 atoms, in the final recognition rate.

28 This set of experiments was conducted using a balanced set of training and val-  
29 idation data. For that, subsets comprised by 4,000 and 1,000 samples for each one  
30 of the classes taken from the training partition were randomly chosen. Hence, new  
31 training and validation matrices  $\mathbf{X}_{trn}$  and  $\mathbf{X}_{val}$  composed by 40,000 and 10,000 sam-  
32 ples, respectively, were built. Here, it is important to point out that two versions of  
33 both training matrices (of sizes  $256 \times 40,000$  and  $1024 \times 40,000$ ) and validation (of  
34 sizes  $256 \times 10,000$  and  $1024 \times 10,000$ ) arise. Also, the whole testing database  $\mathbf{X}_{tst}$  of  
35 sizes  $256 \times 10,000$  and  $1024 \times 10,000$  was left unchanged. In what follows, we shall  
36 denote by DAS-KSVD-256 and DAS-KSVD-1024 structured dictionaries constructed  
37 from samples of dimension 256 and 1024, respectively.

In order to classify the input data into the 10 different classes, a feed-forward Multilayer Perceptron (MLP) neural network with *logsig* activation function was used. In particular this set of experiments begins by varying both the number of discriminant atoms between 20 to 200 (with a step of 20 atoms) and the number of neurons in the hidden layer between 100 to 1000 (with a step of 100 neurons). Left and right sides of Figure 5 show the classification results obtained by DAS-KSVD-256 and DAS-KSVD-1024, respectively. The curves in color blue show averaged accuracy percentages obtained by taken into account a fixed number of discriminant atoms (from 20 to 200) while allowing the number of neurons in the hidden layer to change. Also, the 95% confidence intervals are also included (blue-highlighted regions). In both cases, results show that increasing the structured dictionary size, i.e. the number of discriminant atoms, allows for improving the system performance. As it can be observed, although DAS-KSVD-256 and DAS-KSVD-1024 performs similarly, DAS-KSVD-1024 reached significantly better accuracy percentages than DAS-KSVD-256.

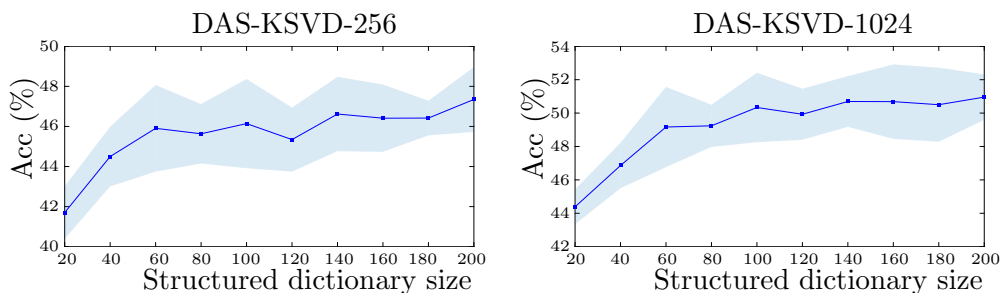


Figure 5: Results achieved by DAS-KSVD-256 (left) and DAS-KSVD-1024 (right) taken into account different number of discriminant atoms while varying the number of neurons in the hidden layer from 100 to 1000.

As can be observed in Figure 5, the best recognition rates were accomplished with structured dictionaries comprised of 200 discriminant atoms. Additionally, it was found that MLP neural networks composed by 1000 neurons in its hidden layer yield in the best accuracy rate of our system. Table 3 shows a comparative summary of the classification results yielded by the proposed DAS-KSVD method and those achieved by both the standard KSVD method and the discriminant-based LC-KSVD2 method, all of them in their both versions (256 and 1024). As can be observed, DAS-KSVD outperforms all other evaluated approaches in their two versions: 256 and 1024. In particular, it was found that applying DAS-KSVD-1024, the classifier yielded an average accuracy rate of 54.4%. Also, the method leading to the second largest performance is LC-KSVD2-1024. When applying such a method, the classifier achieved an average accuracy rate of 51.7%.

Table 3: Test recognition accuracy on CIFAR-10 database using a MLP-200-1000-10 with atoms of dimension 256 and 1024.

Method	Acc (%)
DAS-KSVD-1024	<b>54.4</b>
DAS-KSVD-256	48.8
KSVD-1024 [9]	49.9
KSVD-256 [9]	45.0
LC-KSVD2-1024 [15]	51.7
LC-KSVD2-256 [15]	48.4

## 6. Conclusions

In this work, both a new discriminant measure and a novel method for learning structured dictionaries for multi-class classification problems were introduced. This new measure is capable of efficiently quantifying the degree of discriminability of each one of the atoms in a particular dictionary. The use of such a measure gave rise to what we called the Discriminant Atom Selection KSVD (DAS-KSVD) method for structured dictionary learning. The method was tested with two widely used databases for handwritten digit recognition and for object recognition. The classification performance of DAS-KSVD was also compared with three state-of-the-art classification methods. Experimental results showed that DAS-KSVD significantly outperforms the other three methods achieving good recognition rates and additionally, reducing the computational cost of the classifier.

Clearly, there is much further room for improvements. In particular, future research lines include more analysis of the combined discriminant measure as well as the study of its properties and the exploration of new deep structures.

## 7. Acknowledgments

All acknowledgments will be placed here.

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# Appendices

## A. Grid search

The grid search method starts by dividing the interval  $[0; 1]$  into segments of length  $\Delta$  and generating different combinations of the parameters  $\alpha$  and  $\beta$  such that  $\alpha + \beta \leq 1$ . This constraint suggests that the boundary of the work space coincides with a right triangle whose vertices are the pair of parameters corresponding to  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . Figure 6 shows an example of the grid search method for three different values of  $\Delta$ . It can be observed that small values of  $\Delta$  entail evaluating a large number of combinations.

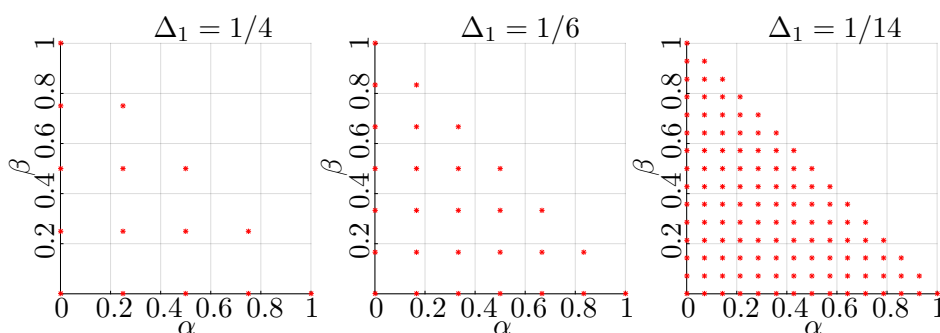


Figure 6: Different possible combinations of weights.

In order to reduce the computational cost, we have performed a grid search of the optimal pair of parameters into two stages. The first one consists of defining and using  $\Delta_1 = 1/6$  in order to locate potential “regions” in the search space where recognition rates are maximized. Also, the second stage takes into account these regions and, moreover, performs a more refined search using  $\Delta_2 = 1/100$ . In that way, each new refined region of search is established by considering all possible pair of parameters complying with  $(\alpha - \alpha^*)^2 + (\beta - \beta^*)^2 \leq (2\Delta_2)^2$  (see Figure 7). This definition coincides with all  $(\alpha, \beta)$  that are inside to a close disc of radius  $2\Delta_2$  centered at  $(\alpha^*, \beta^*)$ .

The most discriminant atoms of  $\Phi$  according to the combined measure  $m_{\alpha, \beta}$  were selected and taken in for building structured dictionaries. As mentioned above, the problem of finding the optimal pair of parameters  $(\alpha^*, \beta^*)$  was solved by applying the grid search method. This search was initially carried out by taking into account an interval length of  $\Delta = 1/6$  which leads to 28 different pair of parameters. Figure 7 shows a summary of the results obtained by applying the grid search method for each one of the four evaluated dictionaries. In particular, we have found that using

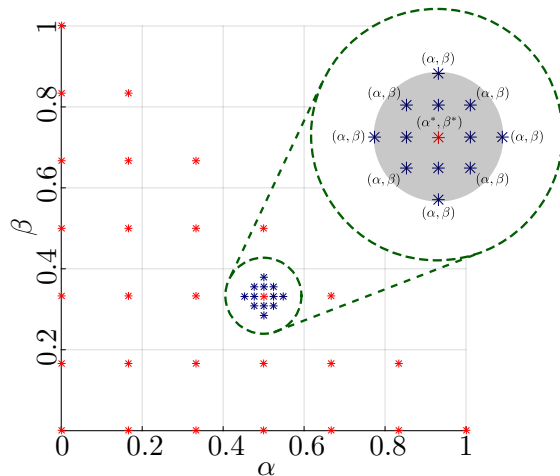


Figure 7: An example illustrating a second stage grid search.

1 structured dictionaries comprised by more than 5 class-related discriminant atoms,  
 2 the MLP neural networks achieved good recognition rates. This figure also shows,  
 3 for each one of the evaluated dictionaries, two highlighted regions denoted by  $R_1$   
 4 and  $R_2$  where recognition rate are maximum. Among all highlighted regions, one  
 5 might think that simultaneous values of  $\alpha$  and  $\beta$  close to zero allow selecting the  
 6 most discriminant atoms of  $\Phi$ . In case of using a structured dictionary comprised  
 7 by 5 discriminant atoms for each one of the classes, we found that search regions  $R_1$   
 8 and  $R_2$  are centered at  $(0.33, 0.17)$  and  $(0.83, 0)$ , respectively, and centered at  $(0, 0)$   
 9 and  $(0.33, 0.17)$ , otherwise.

10 We also analyzed the overall performance (taken over 10 rounds) of the classifier  
 11 for each one of the evaluated dictionaries. As it can be seen in Table 3,  $\Phi_D^{(20)}$  outper-  
 12 forms all the others yielding the maximum (Max) recognition rate. Also, it can be  
 13 seen that small structured dictionary sizes entail low classification rates. This may  
 14 be due to the fact that low dimensional sparse vectors are not capable of capturing  
 15 relevant information for signal classification. Otherwise, if the dimension of such  
 16 vectors increases (from 100 to 200) then significant improvements are observed.

17 The second stage grid search method was successfully applied to each one of the  
 18 tested structured dictionaries. Results have shown that, in this case, no improve-  
 19 ments in the recognition rates were found. Thus, the optimal pair of parameters  $\alpha^*$   
 20 and  $\beta^*$  are the ones found in the first stage. Figure 9 shows the results obtained  
 21 by applying the refined grid search to regions  $R_1$  (left) and  $R_2$  (right) correspond-  
 22 ing to the structured dictionary  $\Phi_D^{(20)}$ . It can be clearly seen that the values of  $\alpha = 0$

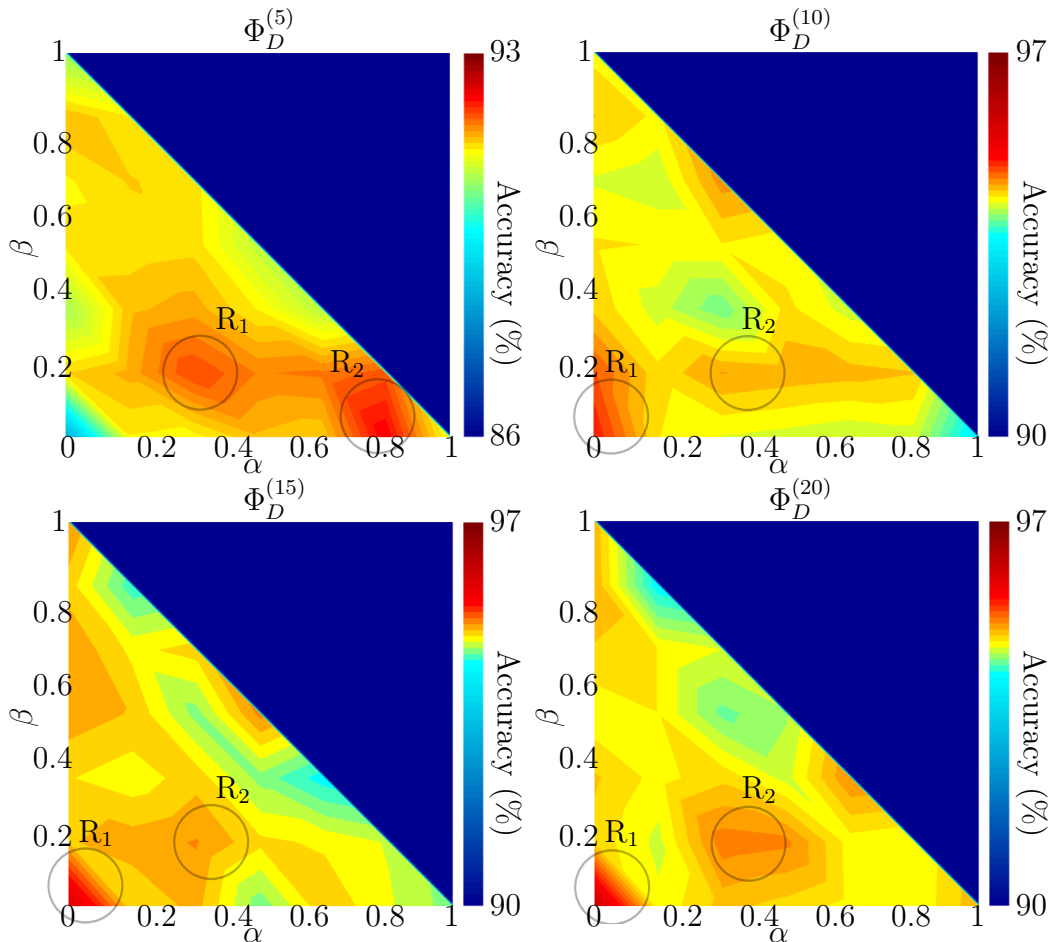


Figure 8: First step grid search results for each one of the evaluated dictionaries of sizes 50 (upper-left), 100 (upper-right), 150 (bottom-left) and 200 (bottom-right).

Table 4: Mean, standard deviation and maximum recognition rates obtained for each one of the evaluated dictionaries.

Dictionary	Classifier	Acc (%)	Max (%)
$\Phi_D^{(5)}$	MLP-50-50-10	91.25 ( $\pm 0.67$ )	92.23
$\Phi_D^{(10)}$	MLP-100-100-10	94.36 ( $\pm 0.27$ )	95.03
$\Phi_D^{(15)}$	MLP-150-150-10	94.74 ( $\pm 0.27$ )	95.90
$\Phi_D^{(20)}$	MLP-200-200-10	94.87 ( $\pm 0.33$ )	<b>96.20</b>

and  $\beta = 0$  suggest that the most discriminant atoms of a particular dictionary  $\Phi$  are 1

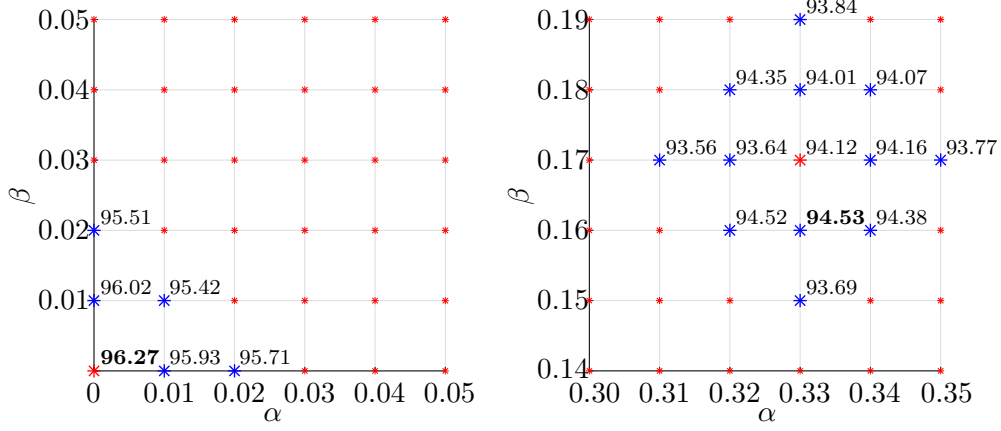


Figure 9: A second stage grid search taken into account regions  $R_1$  (left) and  $R_2$  (right) of  $\Phi_D^{(20)}$ .

- 1 not only those more frequently used for signal representation, but also the ones that
- 2 minimize the total signal representation error. This imply that using only the third
- 3 term of the proposed combined measure, we ensure finding the most discriminant
- 4 atoms of a given dictionary.