Proceedings of the European Control Conference 2007
Kos, Greece, July 2-5, 2007

Polynomial Approach to Nonlinear Predictive GMV Control

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Abstract—**A simple approach is described for the design of nonlinear predictive controllers. The Nonlinear Predictive Generalized Minimum Variance (NPGMV) control algorithm is introduced for nonlinear discrete-time multivariable systems. The system is represented by a combination of a stable nonlinear subsystem where no structure is assumed and a linear subsystem that may be unstable and modelled in polynomial matrix form. The multi-step predictive control cost index to be minimised involves both weighted error and control signal costing terms. The solution for the control law is derived in the time-domain using a very nonlinear operator model of the process. The controller includes an internal nonlinear model of the process but because of the assumed structure of the system, that has a linear disturbance model, the polynomial equations for the predictor are linear.**

I. INTRODUCTION

The aim is to introduce a relatively simple controller for nonlinear systems and one that has some of the advantages of the popular polynomial based *Generalised Predictive Control* (*GPC*) algorithms. It is well known that nonlinear systems have more complex behaviour than linear systems including limit cycle responses and chaotic behaviour. The proposed controller does not rely on local linearization and it provides a global optimal control solution.

The linear *model based predictive control* (*MBPC*) approach has been applied very successfully in the process industries, where it has improved the profitability and competitiveness of a production plant. It has been used to improve performance in difficult systems which contain long dead times, time-varying system parameters and multivariable interactions. The most popular predictive control algorithms are *Dynamic Matrix Control* (*DMC*) [1], *Generalized Predictive Control* (*GPC*) [2,3] and the algorithms of *Richalet* [4,5]. The relationship between *LQ* optimal and *predictive control* was explored in [6].

The *GPC* controller was originally obtained in a polynomial system form. The control strategy developed here also builds upon previous results on *Generalised Minimum Variance* (*GMV*) control. A *Nonlinear Generalized Minimum Variance* (*NGMV*) controller was derived recently for nonlinear *model based* multivariable systems. The assumption was made that the plant model could be decomposed into a set of delay terms, a very general nonlinear subsystem that had to be stable and a linear subsystem that could be represented in polynomial matrix or state equation form and include unstable modes. This problem was analysed in [7-9].

Manuscript received Manuscript received March, 30, 2007. We are grateful for the support of the Engineering and Physical Sciences Research Council on the Platform Grant Project N[o] EP/C526422/1.

The major development over the basic *NGMV* control law involves an extension of the *NGMV* cost-index to include future tracking error and control costing terms in a *GPC* type of problem where the linear sub-system of the plant model is represented in polynomial matrix equation form. When the system is linear the controller is equivalent to a *GPC* controller that is used in many applications. There is a rich history [10,11] of relatively recent research on nonlinear predictive control but the proposed approach is somewhat different, since it is closely linked to the NGMV design philosophy.

II. SYSTEM DESCRIPTION

The assumed model for the plant can be severely nonlinear and dynamic and may have a very general form but the disturbance model is chosen to be linear so that relatively simple results are obtained. This is not restrictive, since in many applications the model for the disturbance signal is only an *LTI* approximation. The system shown in Fig. 1 includes the *nonlinear* plant model together with the linear reference, measurement noise and disturbance signals. The signals $v(t)$ *and* $\xi(t)$ are vector zero-mean, independent, white noise signals. The *measurement noise* signal $\{v(t)\}$ has a constant covariance matrix: $R_f = R_f^T \geq 0$ and the *disturbance* white noise source $\{\xi(t)\}$ has an identity covariance matrix.

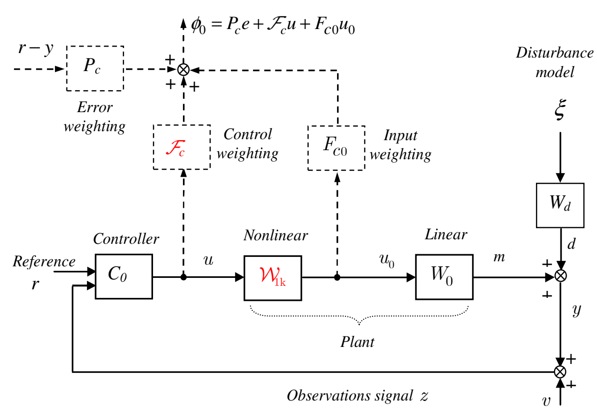

Fig.1: NPGMV 2-Degrees of Freedom Feedback Control

The plant model can have a very general nonlinear operator form, which might involve hard nonlinearities, a state-dependent state-space model, transfer operators or even nonlinear-function look up tables.

Nonlinear Plant: $(\mathcal{W}_1 u)(t) = z^{-k}(\mathcal{W}_{1k} u)(t)$ (1)

ISBN: 978-960-89028-5-5

where $z^{-k}I$ denotes a diagonal matrix of the common delay elements in the output signal paths.

The output of the non-linear subsystem $\mathcal{W}_{1k}$ will be denoted $u_0(t) = (\mathcal{W}_{1k}u)(t)$. For simplicity the nonlinear subsystem: $\mathcal{W}_1$ is assumed to be *finite gain stable* but the linear subsystem, denoted: $W_0 = z^{-k}W_{0k}$, is introduced below and can contain any unstable modes.

A. Linear Subsystem Polynomial Matrix Models

The polynomial matrix system models, for the $(r \times m)$ multivariable system may now be introduced. The sub-systems to be defined are associated with any linear sub-system W_0 in the plant model and the linear disturbance model. The *CARMA* model, representing the linear subsystem of the plant model in *GPC* design, is defined as:

$$A(z^{-1})y(t) = B_{0k}(z^{-1})u_0(t-k) + C_d(z^{-1})\xi(t) \tag{2}$$

where $\{\xi(t)\}$ and the input signal channels in the plant model are assumed to include a *k-steps* ($k \geq 0$) transport delay and $B_0(z^{-1}) = B_{0k}(z^{-1})z^{-k}$. The *delay free* plant transfer of the linear sub-system, referred to above and the *disturbance model* may therefore be defined, without loss of generality, in the left coprime unit-delay operator form:

$$[W_{0k}(z^{-1}) \;\; W_d(z^{-1})] = A(z^{-1})^{-1}[B_{0k}(z^{-1}) \;\; C_d(z^{-1})] \tag{3}$$

For later use introduce a *cost-function weighting model* in left coprime polynomial matrix form, as $P_{c0} = P_{cd}^{-1} P_{cn}$. The weighted output: $y_p(t) = P_{c0}(z^{-1})\,y(t)$:

$$y_p(t) = P_{c0}\, A^{-1}(B_{0k}u_0(t-k) + C_d\xi(t)) \tag{4}$$

The arguments of these polynomial matrices are often omitted. The *power spectrum* for the disturbance and noise signal $f = d + v = W_d\xi + v$ can be computed, noting these are linear, using: $\Phi_{ff} = \Phi_{dd} + \Phi_{vv} = W_d W_d^* + R_f$, where the notation for the adjoint of W_d implies: $W_d^*(z^{-1}) = W_d^T(z)$ and only in this case z represents the z-domain complex number. The *generalized spectral-factor* Y_f may be computed as $Y_f Y_f^* = \Phi_{ff}$, where $Y_f = A^{-1}D_f$. Assume models are such that D_f is a *strictly Schur* polynomial matrix that satisfies:

$$D_f D_f^* = C_d C_d^* + A R_f A^* \tag{5}$$

The model for the disturbance signal is linear, which is an assumption that does not affect stability properties. It is well known that the signal: $f = d + v$ may be modelled in innovations signal form as: $f(t) = Y_f\,\varepsilon(t)$, where $Y_f = A^{-1}D_f$ is defined by the spectral factorisation (5) and $\{\varepsilon(t)\}$ denotes a white noise signal of zero-mean and identity covariance matrix [12]. The system description may be assumed to be such that D_f is *strictly Schur*. The *observations signal* may therefore be written, using (2) and the innovations signal model, as:

$$z(t) = A^{-1}B_{0k}u_0(t-k) + Y_f\varepsilon(t) \tag{6}$$

Define the right coprime model for the weighted factor:

$$P_{c0}(z^{-1})Y_f(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1}) \tag{7}$$

The *weighted observations signal*: $z_p(t) = P_{c0}(z^{-1})z(t)$ or

$$z_p(t) = P_{c0}W_{0k}u_0(t-k) + D_{fp}A_f^{-1}\varepsilon(t) \tag{8}$$

B. Optimal Linear Predictor Problem

The solution of the optimal control problem requires the introduction of a least squares predictor. This enables the inferred output y at times: $t+k+1$, $t+k+2$,. to be calculated (assuming that the disturbance at future times is null). The prediction error cost-function to be minimised,

$$J = E\{\tilde{y}_p(t+j\,|\,t)^2 \tag{9}$$

The estimation error:

$$\tilde{y}_p(t+j\,|\,t) = y_p(t+j) - \hat{y}_p(t+j\,|\,t) \tag{10}$$

and $\hat{y}_p(t+j\,|\,t)$ defines the predicted value of $y_p(t)$ at a time *j* steps ahead. The following *Diophantine equation* must be solved for the solution: (E_j, H_j), E_j smallest degree:

$$E_j(z^{-1})A_f(z^{-1}) + z^{-j-k}H_j(z^{-1}) = D_{fp}(z^{-1}) \tag{11}$$

This equation may be written in the transfer operator form:

$$E_j(z^{-1}) + z^{-j-k}H_j(z^{-1})A_f^{-1}(z^{-1}) = D_{fp}(z^{-1})A_f^{-1}(z^{-1}) \tag{12}$$

Weighted Observations: Substituting from (6), (8), (11):

$$\begin{aligned} z_p(t) &= P_{c0}W_{0k}u_0(t-k) + E_j\varepsilon(t) \\ &+ z^{-j-k}H_jA_f^{-1}\left(Y_f^{-1}z(t) - D_f^{-1}B_{0k}u_0(t-k)\right) \\ &= E_j\varepsilon(t) + z^{-j-k}H_jD_{fp}^{-1}P_{c0}z(t) \\ &+ \left(P_{c0}A^{-1}B_{0k} - z^{-j-k}H_jA_f^{-1}D_f^{-1}B_{0k}\right)u_0(t-k) \end{aligned}$$

Weighted Output Signal: To obtain the expression for the weighted output note $z_p(t) = P_{c0}z(t) = y_p(t) + v_p(t)$, where $y_p(t) = P_{c0}y(t)$ and $v_p(t) = P_{c0}v(t)$. Thence,

$$\begin{aligned} z_p(t) &= E_j\varepsilon(t) + z^{-j-k}H_jD_{fp}^{-1}z_p(t) \\ &+ \left(P_{c0}Y_fA_f - z^{-j-k}H_j\right)A_f^{-1}D_f^{-1}B_{0k}u_0(t-k) \end{aligned} \tag{13}$$

but from (7): $P_{c0} Y_f A_f = D_{fp}$ and from (12) and (13):

$$y_p(t) = E_j\varepsilon(t) - v_p(t) + z^{-j-k} H_j D_{fp}^{-1} z_p(t)$$
$$+\left(D_{fp} - z^{-j-k} H_j\right) A_f^{-1} D_f^{-1} B_{0k} u_0(t-k)$$

Thence, using (11), the $j+k$ *steps ahead* weighted output:

$$y_p(t+j+k) = E_j\varepsilon(t+j+k) - v_p(t+j+k) + H_j D_{fp}^{-1} z_p(t)$$
$$+E_j D_f^{-1} B_{0k} u_0(t+j) \tag{14}$$

To simplify the equations define the right coprime model:

$$B_{1k}(z^{-1}) D_{f1}^{-1}(z^{-1}) = D_f^{-1}(z^{-1}) B_{0k}(z^{-1}) \tag{15}$$

Let signal: $u_f(t) = D_{f1}^{-1}(z^{-1}) u_0(t)$, then (14) is written as:

$$y_p(t+j+k) = \left(E_j(z^{-1})\varepsilon(t+j+k) - v_p(t+j+k)\right)$$
$$+\left[H_j(z^{-1}) D_{fp}^{-1}(z^{-1}) z_p(t) + E_j(z^{-1}) B_{1k}(z^{-1}) u_f(t+j)\right] \tag{16}$$

The maximum degree of the polynomial matrix E_j is $j+k-1$ and hence the noise components in $E_j\varepsilon(t+j+k)$ includes: $\varepsilon(t+j+k)$,..., $\varepsilon(t+1)$, which are at future times.

C. Derivation of the Predictor

Consider the case where the measurement noise term $v(t)$ is zero. The optimal predictor of the output at time $t+j+k$, given observations up to time *t*, can now be derived. The observations, up to time *t* are known and the future values of the control inputs: $u_0(t)$,.., $u_0(t+j)$ used in the predictor are computed at time *t*, and hence the future control input is independent of the future disturbance and noise sequence. It follows that the expected value of the square [.] and round (.) bracketed terms in equation (16) must be zero. The optimal predictor to minimise the cost-function: (9), given that the cross terms in the cost are null, then follows, from (16) as:

$$\hat{y}_p(t+j+k \mid t) = \left[H_j D_{fp}^{-1} z_p(t) + E_j B_{1k} u_f(t+j)\right] \tag{17}$$

Consider the case where the measurement noise is non-zero then the weighted noise $v_p(t+j+k) = P_{c0}(z^{-1}) v(t+j+k)$. If the weighting $P_{c0}(z^{-1})$ is a constant, which is usual in *GPC* control, or if it is assumed a polynomial matrix of degree: $j+k-1$, then $v_p(t+j+k)$ is only dependent on future white measurement noise terms and the expected value of such a term and the square bracketed terms in equation (16) must be zero. The optimal predictor is therefore again given by (17) and the *prediction error*:

$$\tilde{y}_p(t+j+k \mid t) = \left(E_j\varepsilon(t+j+k) - v_p(t+j+k)\right) \tag{18}$$

A second Diophantine equation may be introduced to break up the term: $E_j(z^{-1}) B_{1k}(z^{-1})$ into a part with a *j+1* delay and part depending on $D_{f1}(z^{-1})$ ($u_f(t) = D_{f1}^{-1} u_0(t)$). Thus, for $j \geq 0$, introduce an equation, which has solution $(G_j,\ S_j)$, of *smallest degree* for G_j:

$$G_j(z^{-1}) D_{f1}(z^{-1}) + z^{-j-1} S_j(z^{-1}) = E_j(z^{-1}) B_{1k}(z^{-1}) \tag{19}$$

Thus, $\deg(G_j(z^{-1})) = j$, where $G_j(z^{-1})$ will be written as: $G_j(z^{-1}) = g_0 + g_1 z^{-1} + \cdots + g_j z^{-j}$. The prediction equation, from (17), may now be obtained (for $j \geq 0$) as:

$$\hat{y}_p(t+j+k \mid t) = H_j D_{fp}^{-1} z_p(t) + (G_j D_{f1} + z^{-j-1} S_j) u_f(t+j)$$
$$= H_j D_{fp}^{-1} z_p(t) + G_j u_0(t+j) + S_j u_f(t-1) \tag{20}$$

The degree of $G_j(z^{-1})$ is *j* and the second term in (20) therefore involves future inputs. Define the signal: $f_j(t)$ as:

$$f_j(t) = H_j(z^{-1}) D_{fp}^{-1}(z^{-1}) z_p(t) + S_j(z^{-1}) u_f(t-1) \tag{21}$$

Thus, the *predicted weighted output* (20), for $j \geq 0$,

$$\hat{y}_p(t+j+k \mid t) = G_j(z^{-1}) u_0(t+j) + f_j(t) \tag{22}$$

D. Matrix Representation of the Prediction Equations

The future weighted outputs are to be predicted in the following section for inputs computed in the interval: $\tau\ \varepsilon\ [t, t+N]$ where $N \geq 0$. The equation (22) may therefore be used to obtain the following vector equation for the weighted output at $N+1$ future values of time:

$$\begin{bmatrix} \hat{y}_p(t+k \mid t) \\ \hat{y}_p(t+1+k \mid t) \\ \vdots \\ \\ \hat{y}_p(t+N+k \mid t) \end{bmatrix} = \begin{bmatrix} g_0 & 0 & \cdots & 0 & 0 \\ g_1 & g_0 & 0 & \cdots & 0 \\ \vdots & g_1 & g_0 & & \vdots \\ & \vdots & & \ddots & \\ g_N & g_{N-1} & \cdots & & g_0 \end{bmatrix} \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ \\ u_0(t+N) \end{bmatrix} + \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ \\ f_N(t) \end{bmatrix} \tag{23}$$

Introducing an obvious definition of terms for the matrices in (23) the predicted weighted outputs may be written as:

$$\hat{Y}_{t+k,N} = G_N U_{t,N}^0 + F_{t,N} \tag{24}$$

The vector $F_{t,N}$ may now be found using (21), as:

$$F_{t,N} = \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_N(t) \end{bmatrix} = \begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \\ \vdots \\ H_N(z^{-1}) \end{bmatrix} D_{fp}^{-1}(z^{-1}) z_p(t) + \begin{bmatrix} S_0(t) \\ S_1(t) \\ \vdots \\ S_N(t) \end{bmatrix} u_f(t-1)$$
$$= H_{NZ}(z^{-1}) z_p(t) + S_{NZ}(z^{-1}) u_f(t-1) \tag{25}$$

$$\tilde{Y}_{t+k,N} = \begin{bmatrix} e_0\varepsilon(t+k)+\ldots+e_{k-1}\varepsilon(t+1)-v_p(t+k) \\ e_0\varepsilon(t+1+k)+\ldots+e_k\varepsilon(t+1)-v_p(t+1+k) \\ \vdots \\ e_0\varepsilon(t+N+k)+\ldots+e_{N+k-1}\varepsilon(t+1)-v_p(t+N+k) \end{bmatrix} \quad (26)$$

It is reasonable to assume in many applications that the future variations of the set-point or reference signal $\{r(t)\}$ are predetermined, over a fixed future horizon of *N* steps. The weighted reference is assumed to include the error weighting: $r_p(t) = P_{c0}(z^{-1})r(t)$. The vectors of *weighted* reference, output and input signals may be defined as:

$$R_{t,N} = \begin{bmatrix} r_p(t) \\ r_p(t+1) \\ \vdots \\ r_p(t+N) \end{bmatrix}, \quad Y_{t,N} = \begin{bmatrix} y_p(t) \\ y_p(t+1) \\ \vdots \\ y_p(t+N) \end{bmatrix}, \quad U^0_{t,N} = \begin{bmatrix} u_0(t) \\ u_0(t+1) \\ \vdots \\ u_0(t+N) \end{bmatrix} \quad (27)$$

The *k steps-ahead* future weighted outputs can be written in vector terms $Y_{t+k,N} = \hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}$ and the *future tracking error*, that includes any dynamic error weighting, may therefore be written in vector form as:

$$E_{t+k,N} = R_{t+k,N} - Y_{t+k,N} = R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}) \quad (28)$$

The vector of predicted signals: $\hat{Y}_{t+k,N}$ in (28) and the prediction error: $\tilde{Y}_{t,N}$ are orthogonal.

III. Review of Generalised Predictive Control

A review of the derivation of the *GPC* controller is provided below where the input will be taken to be that for the linear sub-system (u_0), since it also provides results that are needed for the definition of the nonlinear problem of interest. Only the main points in the solution are summarised. The *GPC* performance index to be minimised:

$$J = E\{\sum_{j=0}^{N} e_p(t+j+k)^T e_p(t+j+k) + \lambda_j^2 u_0(t+j)^T u_0(t+j)) \big| t\} \quad (29)$$

where $E\{.|t\}$ denotes the conditional expectation, conditioned on measurements up to time *t*; λ_j denotes a scalar control signal weighting factor and the vector of future *weighted* error signal values $e_p(t+j+k) = P_{c0}(z^{-1})(r(t+j+k) - y(t+j+k))$.

GPC Performance Index: The multi-step cost-function may therefore be written, in a more concise vector form, as:

$$J = E\{J_t\} = E\{(R_{t+k,N} - Y_{t+k,N})^T(R_{t+k,N} - Y_{t+k,N}) + U^{0T}_{t,N}\Lambda_N^2 U^0_{t,N} \mid t\} \quad (30)$$

Introducing the optimal predictor, using (28) and (30),

$$J = E\{(R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}))^T(R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N})) + \\ + U^{0T}_{t,N}\Lambda_N^2 U^0_{t,N} \mid t\} \quad (31)$$

where the cost weightings on the future inputs u_0 are written as: $\Lambda_N^2 = diag\{\lambda_0^2, \lambda_1^2, \ldots, \lambda_N^2\}$.

E. GPC Optimal Control Solution

The terms in the performance index can then be simplified by first noting the prediction errors in $\tilde{Y}_{t+k,N}$ depends on future values of the signal $\{\varepsilon(t)\}$, which are assumed to be independent of future controls. The estimate: $\hat{Y}_{t+k,N}$ is therefore orthogonal to the estimation error: $\tilde{Y}_{t+k,N}$. In addition observe that the future reference or set-point trajectory: $R_{t+k,N}$ is assumed to be a known signal over the *N+1* steps. The vector/matrix form of the cost expression:

$$J = (R_{t+k,N} - \hat{Y}_{t+k,N})^T(R_{t+k,N} - \hat{Y}_{t+k,N}) + U^{0T}_{t,N}\Lambda_N^2 U^0_{t,N} + J_0 \quad (32)$$

where the cost term: $J_0 = E\{\tilde{Y}^T_{t+k,N}\tilde{Y}_{t+k,N} \mid t\}$ is independent of the control action. Substituting (24) into (32) obtain:

$$J = (R_{t+k,N} - (G_N U^0_{t,N} + F_{t,N}))^T(R_{t+k,N} - (G_N U^0_{t,N} + F_{t,N})) + \\ + U^{0T}_{t,N}\Lambda_N^2 U^0_{t,N} + J_0$$

Let $\tilde{R}_{t+k,N} = R_{t+k,N} - F_{t,N}$ and substituting:

$$\begin{aligned} J &= (\tilde{R}_{t+k,N} - G_N U^0_{t,N})^T(\tilde{R}_{t+k,N} - G_N U^0_{t,N}) + U^{0T}_{t,N}\Lambda_N^2 U^0_{t,N} + J_0 \\ &= \tilde{R}^T_{t+k,N}\tilde{R}_{t+k,N} - U^{0T}_{t,N}G_N^T\tilde{R}_{t+k,N} - \tilde{R}^T_{t+k,N}G_N U^0_{t,N} + \\ &\quad + U^{0T}_{t,N}\left(G_N^T G_N + \Lambda_N^2\right)U^0_{t,N} + J_0 \end{aligned} \quad (33)$$

The procedure for minimising this conditional cost function term, is almost identical to that when the signals are deterministic. That is, the gradient of the cost-function must be set to zero, to obtain the vector of future optimal controls. Note that the J_0 term is independent of the control action and a perturbation and gradient calculation may be applied. Setting the gradient to zero gives the vector of *GPC future optimal control signals* as:

$$U^0_{t,N} = \left(G_N^T G_N + \Lambda_N^2\right)^{-1} G_N^T\left(R_{t+k,N} - F_{t,N}\right) \quad (34)$$

The *GPC* optimal control signal at time *t* is based on the *receding horizon* principle [13] and the optimal control is taken as the first element in the vector: $U^0_{t,N}$.

F. Equivalent Cost Minimisation Problem

It is now shown that the above problem is equivalent to a special cost minimisation control problem which is needed to motivate the *NPGMV* problem later. Let the constant positive definite, real symmetric matrix be factorised as:

$$Y^T Y = G_N^T G_N + \Lambda_N^2 \quad (35)$$

Then observe that by *completing the squares* in equation (33) the cost-function may be written as:

$$J = \tilde{R}^T_{t+k,N}\tilde{R}_{t+k,N} - U^{0T}_{t,N}G^T_N\tilde{R}_{t+k,N} - \tilde{R}^T_{t+k,N}G_N U^0_{t,N}$$
$$+U^{0T}_{t,N}Y^TYU^0_{t,N} + J_0$$
$$= \left(\tilde{R}^T_{t+k,N}G_N Y^{-1} - U^{0T}_{t+k,N}Y^T\right)\left(Y^{-T}G^T_N\tilde{R}_{t+k,N} - YU^0_{t,N}\right)$$
$$+\tilde{R}^T_{t+k,N}(I - G_N Y^{-1}Y^{-T}G^T_N)\tilde{R}_{t+k,N} + J_0$$

That is, the cost may be written in an equivalent form as:

$$J = \hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} + J_{10}(t) \tag{36}$$

where

$$\hat{\Phi}_{t+k,N} = Y^{-T}G^T_N(R_{t+k,N} - F_{t,N}) - YU^0_{t,N} \tag{37}$$

The terms that are independent of the control action may be written as: $J_{10}(t) = J_0 + J_1(t)$ where

$$J_1(t) = \tilde{R}^T_{t+k,N}(I - G_N Y^{-1}Y^{-T}G^T_N)\tilde{R}_{t+k,N} \tag{38}$$

Since the last term $J_{10}(t)$ in equation (36) does not depend upon control action the optimal control is found by setting the first term to zero, giving the same optimal control as defined in equation: (34). It follows that the *GMV* optimal controller for the above linear system is the same as the controller to minimise the Euclidean norm of the *deterministic signal*: $\hat{\Phi}_{t+k,N}$, defined in (37).

G. Modified Cost-Function

The above discussion motivates the definition of a new *multi-step minimum variance cost* problem that has the same solution for the optimal controller. This is needed before the nonlinear problem of real interest can be considered. There are some mathematical preliminaries and the result is then presented. Consider first a new signal to be minimised involving a weighted sum of error and input signals of form:

$$\phi = P_{C0}(z^{-1})(r(t) - y(t)) + F_{C0}u_0 \tag{39}$$

The vector of future values of this signal, may be written as:

$$\Phi_{t,N} = P_{CN}E_{t,N} + F^0_{CN}U^0_{t,N} \tag{40}$$

Now introduce cost-function weightings, using the original *GMV* weightings, but to have the constant matrix form:

$$P_{CN} = Y^{-T}G^T_N \text{ and } F^0_{CN} = -Y^{-T}\Lambda^2_N \tag{41}$$

The reason for this choice of cost terms becomes apparent in the Theorem 3.1 below. Motivated by the preceding analysis define a *minimum variance multi-step* cost-function,

$$\tilde{J} = E\{\tilde{J}_t\} = E\{\Phi^T_{t+k,N}\Phi_{t+k,N} \mid t\} \tag{42}$$

Predicting forward *k-steps*:

$$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F^0_{CN}U^0_{t,N} \tag{43}$$

Now consider the signal: $\Phi_{t+k,N}$ and substitute for the vector of outputs: $Y_{t+k,N} = \hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N}$. Then from (43) obtain:

$$\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - (\hat{Y}_{t+k,N} + \tilde{Y}_{t+k,N})) + F^0_{CN}U^0_{t,N} \tag{44}$$

This expression may be written as:

$$\Phi_{t+k,N} = \hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N} \tag{45}$$

where the predicted signal:
$\hat{\Phi}_{t+k,N} = \left(P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F^0_{CN}U^0_{t,N}\right)$ and the prediction error:

$$\tilde{\Phi}_{t+k,N} = -P_{CN}\tilde{Y}_{t+k,N} \tag{46}$$

The performance index (42) may therefore be simplified as: $\tilde{J} = E\{\tilde{J}_t\} = E\{(\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N})^T(\hat{\Phi}_{t+k,N} + \tilde{\Phi}_{t+k,N}) \mid t\}$.
The terms in the performance index (42) can again be simplified, recalling the optimal estimate: $\hat{Y}_{t+k,N}$ and the estimation error: $\tilde{Y}_{t+k,N}$ are orthogonal and the future reference or set-point trajectory: $R_{t+k,N}$ is a known signal. Thus expanding obtain:

$$\tilde{J} = E\{\hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} \mid t\} + E\{\hat{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\} +$$
$$+E\{\tilde{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} \mid t\} + E\{\tilde{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\}$$
$$= \hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} + E\{\tilde{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\} \tag{47}$$

Thence, the cost-function may be written as:

$$\tilde{J}(t) = \hat{\Phi}^T_{t+k,N}\hat{\Phi}_{t+k,N} + \tilde{J}_1(t) \tag{48}$$

The cost term independent of control may be written as:

$$\tilde{J}_1(t) = E\{\tilde{\Phi}^T_{t+k,N}\tilde{\Phi}_{t+k,N} \mid t\} = E\{\tilde{Y}^T_{t+k,N}P^T_{CN}P_{CN}\tilde{Y}_{t+k,N} \mid t\} \tag{49}$$

Now simplify the vector of predicted signals: $\hat{\Phi}_{t+k,N}$ by substituting for $\hat{Y}_{t+k,N}$ from (24) and using (35) and (41):

$$\hat{\Phi}_{t+k,N} = P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F^0_{CN}U^0_{t,N}$$
$$= P_{CN}R_{t+k,N} - P_{CN}(G_N U^0_{t,N} + F_{t,N}) + F^0_{CN}U^0_{t,N}$$

Substituting from (35) obtain,

$$\hat{\Phi}_{t+k,N} = P_{CN}(R_{t+k,N} - F_{t,N}) - YU^0_{t,N} \tag{50}$$

From a similar argument to that in the previous section the optimal multi-step minimum variance control sets the first *squared term* in (48) to zero. The optimal control follows: $U^0_{t,N} = Y^{-1}P_{CN}(R_{t+k,N} - F_{t,N})$. This vector of future optimal controls is the same as the vector of future *GPC* controls (34) and these results are summarised below.

Theorem 3.1: Equivalent Minimum Variance Problem
Consider the minimisation of the *GPC* cost index (29) for the system and assumptions introduced in §2, where the

nonlinear subsystem: $\mathcal{W}_{1k}' = I$ and the vector of optimal *GPC* controls is given by (34). Redefine the cost index to have a multi-step variance form (42):

$\tilde{J}(t) = E\{\Phi_{t+k,N}^T \Phi_{t+k,N} \mid t\}$,

where $\Phi_{t+k,N} = P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0$ and the cost weightings: $P_{CN} = Y^{-T} G_N^T$ and $F_{CN}^0 = -Y^{-T}\Lambda_N^2$.

Then the vector of future optimal controls is identical to the *GPC* controls defined in (34). ■

Solution: The proof follows by simply collecting together the above results. ■

IV. The NPGMV Control Problem

The *Nonlinear Predictive Generalised Minimum Variance* (*NPGMV*) control problem of real interest is now considered. The actual input to the system is of course the control signal $u(t)$**,** shown in Fig. 1, rather than the input to the linear sub-system: u_0. The cost-function for the nonlinear control problem of interest must therefore include an additional control signal costing term, although the costing on the intermediate signal $u_0(t)$ can be retained to examine limiting cases and to provide a useful *actuator output* costing. If the smallest delay in each output channel of the plant is of magnitude k *-steps* this implies that the control signal *t* affects the output at least k *-steps* later. For this reason the control signal costing should be defined as:

$$(\mathcal{F}_c u)(t) = z^{-k}(\mathcal{F}_{ck} u)(t) \tag{51}$$

Typically this weighting on the nonlinear sub-system input will be a *linear dynamic operator* but it may also be chosen to introduce an anti-windup capability [9]. The control weighting operator $\mathcal{F}_{ck}$ will be assumed to be full rank and invertible. Thus, consider a new signal to be minimised, involving a weighted sum of error, input and control signals:

$$\phi_0(t) = P_c e(t) + F_{c0} u_0(t) + (\mathcal{F}_c u)(t) \tag{52}$$

In analogy with the previous *GPC* problem a *multi-step cost index* may now be defined that is an extension of (42).

$$J_p = E\{\Phi_{t+k,N}^{0T} \Phi_{t+k,N}^0 \mid t\} \tag{53}$$

The signal $\Phi_{t+k,N}^0$ is therefore extended to include the additional future control signal costing term:

$$\begin{aligned}\Phi_{t+k,N}^0 &= P_{CN} E_{t+k,N} + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N}) \\ &= P_{CN}(R_{t+k,N} - Y_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N})\end{aligned} \tag{54}$$

The NL function $\mathcal{F}_{ck,N} U_{t,N}$ will have the diagonal form:

$$\begin{aligned}&(\mathcal{F}_{ck,N} U_{t,N}) = \\ &diag\{(\mathcal{F}_{ck} u)(t),\, (\mathcal{F}_{ck} u)(t+1), \ldots,\, (\mathcal{F}_{ck} u)(t+N)\}\end{aligned} \tag{55}$$

and $U_{t,N}^0 = (\mathcal{W}_{1k,N} U_{t,N})$, where $\mathcal{W}_{1k,N}$ has a block form:

$$\begin{aligned}(\mathcal{W}_{1k,N} U_{t,N}) &= diag\{\mathcal{W}_{1k}', \mathcal{W}_{1k}', \ldots, \mathcal{W}_{1k}'\} U_{t,N} \\ &= [(\mathcal{W}_{1k}' u)(t)^T, \ldots, (\mathcal{W}_{1k}' u)(t+N)^T]^T\end{aligned} \tag{56}$$

H. The NPGMV Control Solution

The solution follows from very similar steps to those in §3.3 and will therefore be summarised only briefly below. Observe from (43) that $\Phi_{t,N}^0 = \Phi_{t,N} + z^{-k}(\mathcal{F}_{ck,N} U_{t,N})$ and

$\Phi_{t+k,N}^0 = \hat{\Phi}_{t+k,N}^0 + \tilde{\Phi}_{t+k,N}^0$

where $\hat{\Phi}_{t+k,N}^0 = \hat{\Phi}_{t+k,N} + (\mathcal{F}_{ck,N} U_{t,N})$

$$= P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + F_{CN}^0 U_{t,N}^0 + (\mathcal{F}_{ck,N} U_{t,N}) \tag{57}$$

and the estimation error:

$$\tilde{\Phi}_{t+k,N}^0 = \tilde{\Phi}_{t+k,N} = -Y^{-T} G_N^T \tilde{Y}_{t+k,N} \tag{58}$$

The future predicted values in the signal: $\hat{\Phi}_{t+k,N}^0$ involve the estimated weighted outputs $\hat{Y}_{t+k,N}$ orthogonal to $\tilde{Y}_{t+k,N}$. The estimation error is zero mean and the expected value of the product with any known signal is null. The cost follows:

$$\tilde{J}(t) = \hat{\Phi}_{t+k,N}^{0T} \hat{\Phi}_{t+k,N}^0 + \tilde{J}_1(t) \tag{59}$$

where the optimal control sets: $\hat{\Phi}_{t+k,N}^0 = 0$. The condition for optimality has the form:

$$P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) + (\mathcal{F}_{ck,N} + F_{CN}^0 \mathcal{W}_{1k,N}) U_{t,N} = 0 \tag{60}$$

I. The Nonlinear Predictive GMV Control Signal

The future optimal control signals, to minimise the cost index:(59), follows from (60) and satisfies:

$$U_{t,N} = -(\mathcal{F}_{ck,N} - Y^{-T}\Lambda_N^2 \mathcal{W}_{1k,N})^{-1} P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) \tag{61}$$

An alternative solution of equation (60),

$$U_{t,N} = -\mathcal{F}_{ck,N}^{-1}\left(P_{CN}(R_{t+k,N} - \hat{Y}_{t+k,N}) - Y^{-T}\Lambda_N^2 \mathcal{W}_{1k,N} U_{t,N}\right) \tag{62}$$

The optimal predictive control law is clearly nonlinear, since it involves the nonlinear control signal costing term: $\mathcal{F}_{ck,N}$ and the nonlinear model for the plant: $\mathcal{W}_{1k,N}$. Further simplification is possible by substituting from (24) for the estimate $\hat{Y}_{t+k,N}$, since (60) may be written as:

$$P_{CN}\left(R_{t+k,N} - F_{t,N}\right) + (\mathcal{F}_{ck,N} - (P_{CN} G_N - F_{CN}^0)\mathcal{W}_{1k,N}) U_{t,N} = 0$$

and substituting from (41) the condition for optimality:

$$P_{CN}\left(R_{t+k,N} - F_{t,N}\right) + (\mathcal{F}_{ck,N} - Y\mathcal{W}_{1k,N}) U_{t,N} = 0 \tag{63}$$

The two alternative solutions for the *future controls*:

$$U_{t,N} = -(\mathcal{F}_{ck,N} - Y\mathcal{W}_{1k,N})^{-1} P_{CN}(R_{t+k,N} - F_{t,N}) \tag{64}$$

$$U_{t,N} = -\mathcal{F}_{\mathrm{c}k,N}^{-1}\left(P_{CN}(R_{t+k,N} - F_{t,N}) - Y\mathcal{W}_{1\mathrm{k,N}}U_{t,N}\right) \quad (65)$$

The controller again invokes a *receding horizon philosophy*. The assumption needed to ensure closed-loop stability is explained in the analysis of stability in [8].

Theorem 4.1: Nonlinear Predictive GMV Control Law

Consider the linear components of the plant and disturbance model in (2) and the plant model in (1), with input from the nonlinear finite gain stable plant dynamics $\mathcal{W}_{1k}$. Let the dynamic error $P_c(z^{-1})$ and the input weightings $\{\lambda_0,...,\lambda_N\}$ be specified and assume the *control signal cost-function weighting* is defined in the form: $(\mathcal{F}_{\mathrm{c}}u)(t) = (\mathcal{F}_{\mathrm{c}k}u)(t-k)$, where k represents any explicit delay and $\mathcal{F}_{ck}$ is full rank and invertible. The *multi-step predictive control* cost-function to be minimised, involves a sum of future cost terms, defined in vector form:

$$J_p = E\{\Phi^{0T}_{t+k,N}\Phi^{0}_{t+k,N} \mid t\} \quad (66)$$

This signal $\Phi^0_{t+k,N}$ includes the vector of future error, input and control signal costing terms:

$$\Phi^0_{t+k,N} = P_{CN}E_{t+k,N} + F^0_{CN}U^0_{t,N} + (\mathcal{F}_{\mathrm{c}k,N}U_{t,N}) \quad (67)$$

where $P_{CN} = Y^{-T}G_N^T$, $F^0_{CN} = -Y^{-T}\Lambda_N^2$ and $\mathcal{F}_{\mathrm{c}k,N}$ is normally a diagonal control weighting (55). Also define the constant matrix factor Y to satisfy: $Y^TY = G_N^TG_N + \Lambda_N^2$, then the *NPGMV* optimal control law to minimize the variance (66) is given as:

$$U_{t,N} = -(\mathcal{F}_{\mathrm{c}k,N} - Y\mathcal{W}_{1\mathrm{k,N}})^{-1}P_{CN}(R_{t+k,N} - F_{t,N}) \quad (68)$$

The preferred expression for implementation of the *vector of future optimal control signals,* assuming the current control is found using the *receding horizon philosophy,* follows:

$$U_{t,N} = -\mathcal{F}_{\mathrm{c}k,N}^{-1}\left(P_{CN}(R_{t+k,N} - F_{t,N}) - Y\mathcal{W}_{1\mathrm{k,N}}U_{t,N}\right) \quad (69)$$

where the signals $F_{t,N} = H_{NZ}(z^{-1})z(t) + S_{NZ}(z^{-1})u_f(t-1)$ and $u_f(t) = D_{f1}^{-1}(z^{-1})u_0(t)$. ■

Solution: The proof of the *NPGMV* optimal control follows by simply collecting the results in the above section. ■

The two expressions for the NPGMV control signal (68) and (69) lead to the two alternative structures, shown in Figs. 3 and 4, respectively. The second, shown in Fig. 4 shows how the current and future controls may be separated from the full vector of future controls, as explained below.

J. Implementation of the Predictive Optimal Control

A useful partition may be introduced which enables the algorithm to be simplified. The control at time *t* is computed for $N > 0$ from the vector of future controls by introducing:

$$C_{I0} = [I,0,....,0] \quad (70)$$

Thence, $u(t) = [I,0,....,0]\,U_{t,N}$ (71)

To be able to compute the vector of future controls for *t > 0* also introduce the matrix: $C_{0I} = [0 \quad I_N]$ (72)

$$U^f_{t,N} = C_{0I}U_{t,N} = \begin{bmatrix}0 & I_N\end{bmatrix}\begin{bmatrix}u(t)\\ \vdots \\ u(t+N)\end{bmatrix} = \begin{bmatrix}u(t+1)\\ \vdots \\ u(t+N)\end{bmatrix} \quad (73)$$

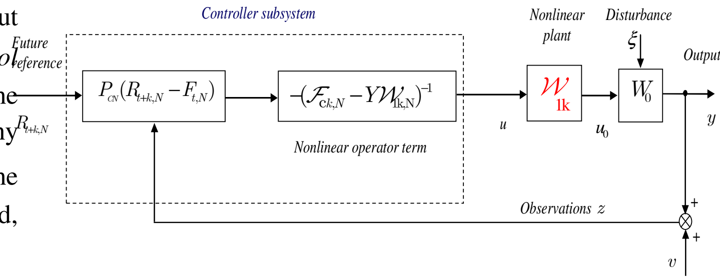

Fig. 3: First Form of NPGMV Controller Structure

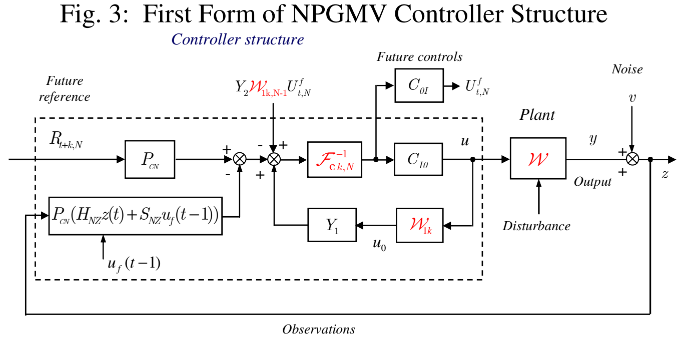

Fig. 4: Second Form of NPGMV Controller

Note from (70), because of the block diagonal structure of the control signal costing $\mathcal{F}_{\mathrm{c}k,N}$, then $C_{I0}\mathcal{F}^{-1}_{\mathrm{c}k,N}$ $= [\mathcal{F}^{-1}_{ck},0,...,0] = \mathcal{F}^{-1}_{ck}C_{I0}$. Using (69) the *optimal control*:

$$u(t) = -\mathcal{F}^{-1}_{ck}C_{I0}\left(P_{CN}(R_{t+k,N} - F_{t,N}) - Y\mathcal{W}_{1\mathrm{k,N}}U_{t,N}\right) \quad (74)$$

The *vector of future controls*, computed at time t,

$$U^f_{t,N} = -C_{0I}\mathcal{F}^{-1}_{\mathrm{c}k,N}\left(P_{CN}(R_{t+k,N} - F_{t,N}) - Y\mathcal{W}_{1\mathrm{k,N}}U_{t,N}\right) \quad (75)$$

Also note from equation (56) that $\mathcal{W}_{1\mathrm{k,N}}U_{t,N}$ may be given as:

$$(\mathcal{W}_{1\mathrm{k,N}}U_{t,N}) = [(\mathcal{W}_{1k}u)(t)^T,...,(\mathcal{W}_{1k}u)(t+N)^T]^T$$
$$= [(\mathcal{W}_{1k}u)(t)^T,\ (\mathcal{W}_{1\mathrm{k,N\text{-}1}}U^f_{t,N})^T]^T \quad (76)$$

Using a related partition, write the matrix Y in the form: $Y = [Y_1 \quad Y_2]$, where Y_1 has m_0 columns, so that:

$$Y\mathcal{W}_{1\mathrm{k,N}}U_{t,N} = [Y_1 \ \ Y_2]\mathcal{W}_{1\mathrm{k,N}}U_{t,N} = Y_1(\mathcal{W}_{1k}u)(t) + Y_2(\mathcal{W}_{1\mathrm{k,N\text{-}1}}U^f_{t,N})$$

Using (74) the current and future *NPGMV* optimal controls:

$$u(t) = -\mathcal{F}^{-1}_{\mathrm{c}k}C_{I0}[P_{CN}(R_{t+k,N} - F_{t,N})$$
$$-Y_1(\mathcal{W}_{1k}u)(t) - Y_2\mathcal{W}_{1\mathrm{k,N\text{-}1}}U^f_{t,N}] \quad (77)$$

$$U^f_{t,N} = -C_{0I}\mathcal{F}^{-1}_{\mathrm{c}k,N}(P_{\scriptscriptstyle CN}(R_{t+k,N} - F_{t,N}) +$$
$$-Y_1(\mathcal{W}_{1\mathrm{k}}u)(t) - Y_2(\mathcal{W}_{1\mathrm{k,N\text{-}1}}U^f_{t,N}) \quad (78)$$

There is a considerable simplification to the structure of the controller if the matrix Y^TY defined in (35), is factorised into a block lower triangular matrix form Y where the first m_0 rows of Y_2 are null. The last terms in (77) is then null, since the term: $C_{\mathrm{I}0}Y_2 = 0$.

Lemma 4.1: NPGMV Control Law Properties

Consider again the system described in §2 and the predictive control problem for the cost index (53) (*N > 0*). The nonlinear plant operator: $\mathcal{W}_{1k}$ is assumed to be *finite gain stable* and for closed-loop stability the operator:

$$\left((Y_{11} + C_{I0}P_{\scriptscriptstyle CN}W_{Nk})\mathcal{W}_{1\mathrm{k}} - \mathcal{F}_{\mathrm{ck}}\right)$$

is assumed to have a stable causal inverse, due to the choice of the weightings that determine the operators: $P_{\scriptscriptstyle CN}$, F^0_{CN} and $\mathcal{F}_{\mathrm{c}k,}$. By partitioning the optimal control to be applied at time *t,* invoking the *receding horizon principle,* the vectors of current and future predicted controls:

$$u(t) = -\mathcal{F}^{-1}_{ck}C_{I0}\left(P_{\scriptscriptstyle CN}(R_{t+k,N} - F_{t,N}) - Y_1(\mathcal{W}_{1\mathrm{k}}u)(t)\right) \quad (79)$$

$$U^f_{t,N} = -C_{0I}\mathcal{F}^{-1}_{\mathrm{c}k,N}(P_{\scriptscriptstyle CN}(R_{t+k,N} - F_{t,N}) - Y_1(\mathcal{W}_{1\mathrm{k}}u)(t) +$$
$$-Y_2(\mathcal{W}_{1\mathrm{k,N\text{-}1}}U^f_{t,N})) \quad (80)$$

where $C_{I0} = [I, 0, \ldots, 0]$, $C_{0I} = \begin{bmatrix}0 & I_N\end{bmatrix}$, and Y is assumed to be factorised into a block lower triangular form $Y = [Y_1\ Y_2]$ with $C_{I0}Y_1 = Y_{11}$. If the error and input cost-function weightings are defined in the *GPC* motivated form: $P_{CN} = Y^{-T}G^T_N$ and $F^0_{CN} = -Y^{-T}\Lambda^2_N$ then for a linear system ($\mathcal{W}_{1k} = I$) the limiting form of the optimal control when $\mathcal{F}_{\mathrm{c}} \to 0$ is identical to a standard *GPC* control law.

Solution: The different forms of the control law for computation follows by collecting the results in the section above and the relationship to linear *GPC* control was established from the definitions of the weightings in § 3 and in the Theorem 3.1. The assumption made to ensure closed-loop stability, at the beginning of the theorem, is justified in the analysis of stability in [9]. ■

Remarks: The condition $C_{\mathrm{I}0}Y_2 = 0$ is a valuable property since it ensures future controls do not create an algebraic loop in the calculation of the current control. This also simplifies the computation of current control, as in Fig. 5.

V. CONCLUDING REMARKS

The *Nonlinear Predictive Generalised Minimum Variance* (*NPGMV)* control design problem for a state-space system involved a multi-step predictive control cost-function and provided a method of introducing future set-point information. The predictive controls strategy described is a development of the nonlinear generalised minimum variance (*NGMV*) design method which has been shown to be easy to design and implement. It also has the nice property that if the system is linear then the control can revert to the generalised predictive control design method which is well known and well accepted in industry. That is, the *NPGPC* control design method reduces to that of *GPC* control design when the weight $\mathcal{F}_{\mathrm{c}k}$ tends to zero and the system is linear.

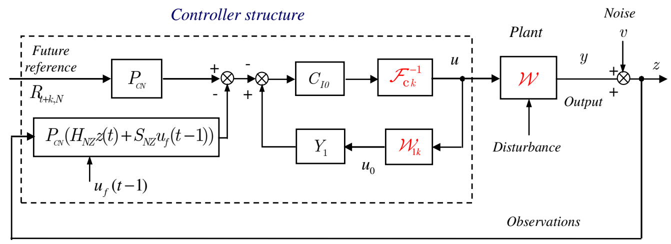

Fig. 5: NPGMV Control Signal Generation

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